E6712: Midterm Exam - Solutions

(Total 100 points)

Given: March 22, 2005

1. Consider the following vector channel detection problem. There are three possible signals:

$$\mathbf{s}_1 = \{\sqrt{1-\alpha}, 0, \sqrt{\alpha}\}\ \mathbf{s}_2 = \{-\sqrt{1-\alpha}/2, \sqrt{3}\sqrt{1-\alpha}/2, \sqrt{\alpha}\}\ \mathbf{s}_3 = \{-\sqrt{1-\alpha}/2, -\sqrt{3}\sqrt{1-\alpha}/2, \sqrt{\alpha}\}$$

The received signal is given by

$$\mathbf{r} = \mathbf{s}_i + \mathbf{n}, \quad i \in \{1, 2, 3\},$$

where $\mathbf{n}$ is a 3-dimensional noise vector whose components are i.i.d. Gaussian RVs with zero mean and variance σ^2. Assume that the prior probabilities of the three possible messages are equal.

(a) *(25 pts)* Consider the case when $\alpha = 0$.

i. Derive an optimum receiver decision rule and draw a block diagram of the optimum receiver.

ii. Draw the signal constellation and decision regions.

iii. Derive an expression for this receiver's error probability, and its Union-Chernoff bound.

(b) *(25 pts)* Consider the case when $\alpha = 1/3$.

i. Derive an optimum receiver decision rule and draw a block diagram of the optimum receiver.

ii. Draw the signal constellation. *Hint:* Compute the angles between the signal vectors; consider a change of coordinates.

iii. Derive an expression for this receiver's error probability, and its Union-Chernoff bound.

(c) *(10 pts)* Consider now transmitting the signals over a uniformly random-phase channel

$$\mathbf{r} = e^{j\theta}\mathbf{s}_i + \mathbf{n}, \quad i \in \{1, 2, 3\}.$$

i. Draw a block diagram of the optimum receiver.

ii. Would you choose the constellation with $\alpha = 0$ or with $\alpha = 1/3$ for this channel? Why?

Solution

(a) $\alpha = 0$:

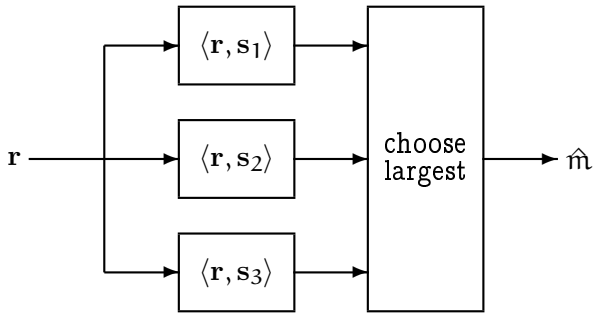
i. The optimal detector finds (*see Lec. 3*)

$$\arg \min_{m \in \{1,2,3\}} \Big[\sum_{j=1}^{3} (r_j - s_{mj})^2\Big],$$

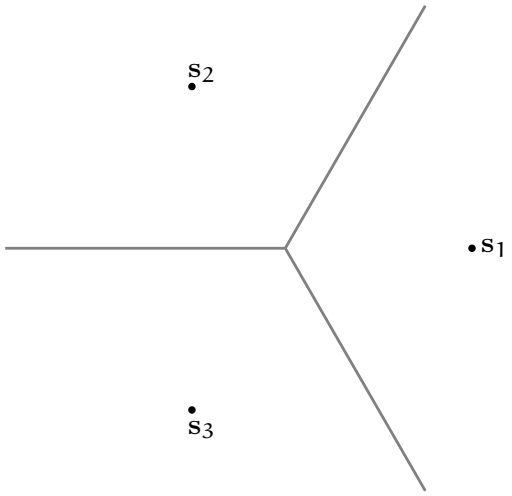
or, equivalently,

$$\arg \max_m \sum_{j=1}^{3} s_{mj} r_j = \arg \max_m \langle \mathbf{s}_m, \mathbf{r} \rangle.$$

A block diagram of the optimum receiver:



ii. Signal constellation with decision regions:



iii. By the genie, union, and Chernoff bound, the probability of error can

$$Q\left(\sqrt{\frac{d_{\min}^2}{4\sigma^2}}\right) < P_e < 2Q\left(\sqrt{\frac{d_{\min}^2}{4\sigma^2}}\right) < \exp\left(-\frac{d_{\min}^2}{8\sigma^2}\right),$$

where $d_{\min}^2 = 3$.

(b) $\alpha = 1/3$:

i. See (a)-i.

ii. Computing angles shows that the constellation consists of three pairwise orthogonal signals.

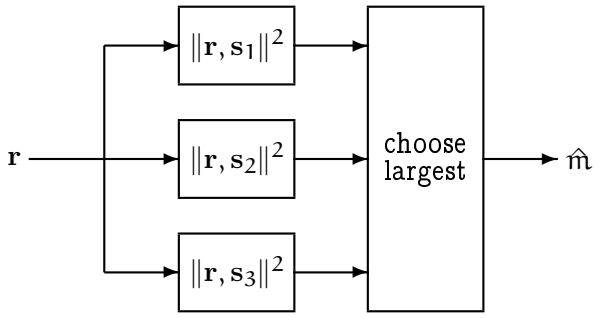
iii. Same as (a)-iii, but $d_{\min}^2 = 2$.

(c) Random phase channel:

i. The optimal detector finds (*see Lec. 5*)

$$\arg\max_m |\langle \mathbf{s}_m, \mathbf{r} \rangle|^2.$$

A block diagram of the optimum receiver:



ii. HW4

2. Consider the following partial response channel:

$$v_k = I_k + I_{k-1} + \eta_k,$$

where $\{\eta_k\}$ is discrete AWGN with zero mean and spectral density $\mathcal{N}_0/2$. The user data $D_k \in \{0,1\}$ is binary, and the channel can take only bipolar signal values $I_k \in \{-1,1\}$.

(a) *(10 pts)* Suppose that user data is directly mapped into the channel sequence as

$$I_k = 2D_k - 1$$

Suppose that $\{I_k\}$ is the sequence $1,-1,1,-1,1,\ldots$, and the MLSE decodes a sequence $\{\widehat{I_k}\}$ with an error event starting at time n_0 and ending at time n_{l-1}. What is the most likely incorrect sequence $\{\widehat{D_k}\}$?

(b) *(10 pts)* Derive a lower bound on the sequence error probability for the MLSE.

(c) *(10 pts)* Suppose that you have to use a symbol-by-symbol detector rather than the MLSE. How would you organize transmission and detection for this channel?

(d) *(10 pts)* Derive a lower bound on the error probability for your receiver.

Solution

(a) The minimum (merged) distance on the bipolar channel with transfer function $(1-D)$ is given by

$$d^2_{<>} = \min_{\varepsilon(D)\neq 0} \|(1+D)\varepsilon(D)\|^2,$$

where the squared norm of a polynomial is defined as the sum of its squared coefficients, and $\varepsilon(D)$ is the polynomial corresponding to a finite length normalized error sequence:

$$\boldsymbol{\varepsilon} = \{\varepsilon_{n_0}, \varepsilon_1, \ldots, \varepsilon_{n_l-1}\} \text{ of length } l < \infty,\ \varepsilon_i \in \{-1,0,1\},\ \varepsilon_{n_0}, \varepsilon_{n_l-1} \neq 0$$

Since, the normalized error sequence corresponding to $\{I_n\}$ and $\{\widehat{I}_n\}$ is

$$\varepsilon_n = \frac{1}{2}(I_n - \widehat{I}_n),$$

we have $\varepsilon_i \in \{0,1\}$, $\varepsilon_{n_0} = 1$, and thus

$$\begin{aligned} d^2_{<>} &= \min_{\varepsilon(D)\neq 0} \|(1+D)\sum_{i=0}^{l-1} \varepsilon_{n_i} D^i\|^2 \\ &= \min_{\varepsilon(D)\neq 0} \|\varepsilon_{n_0} + \sum_{i=1}^{l-1} (\varepsilon_{n_i} + \varepsilon_{n_{i-1}}) D^i + \varepsilon_{n_{l-1}} D^l\|^2 \\ &= \min_{\varepsilon(D)\neq 0} (1 + \sum_{i=1}^{l-1} (\varepsilon_{n_i} + \varepsilon_{n_{i-1}})^2 + 1) \\ &= 2 + \min_{\varepsilon(D)\neq 0} \sum_{i=1}^{l-1} (\varepsilon_{n_i} + \varepsilon_{n_{i-1}})^2 \geq 2, \end{aligned}$$

where the equality holds iff

$$\varepsilon(D) = 1 + \sum_{i=1}^{l-1} (-1)^i D^i.$$

Therefore $\widehat{I}_n = -I_n$, and thus $\{\widehat{D_k}\}$ is the sequence $0,1,0,1,\ldots$

(b) A lower bound to the probability of error (*see Lec. 8*):

$$P_e \geq Q\left(\sqrt{\frac{2d^2}{\mathcal{N}_0} d^2_{<>}}\right) = Q\left(\sqrt{\frac{4}{\mathcal{N}_0}}\right)$$

(c) We can use precoding for the duobinary channel as follows (*see Lec. 7*):

Data sequence $\{D_n\}$

Precoded sequence $P_n = D_n + P_{n-1} \mod 2$

Transmitted sequence $I_n = 2P_n - 1$

Received sequence $B_n = I_n + I_{n-1}$

Decoded sequence: $D_n = \frac{1}{2}B_n + 1 \mod 2$

(d) Note that B_n can take 3 possible values: $\{-2,0,2\}$. Therefore $d^2_{\min} = 4$, and thus

$$P_e \geq Q\left(\sqrt{\frac{d^2_{\min}}{2\mathcal{N}_0}}\right) = Q\left(\sqrt{\frac{2}{\mathcal{N}_0}}\right)$$