EL606: Homework 10 – Solutions

Due date: Dec. 7, 2004

1. *Problem 3, Ch. 10, Cover&Thomas*
   It is clear that the input distribution that maximizes the capacity is $X \sim \mathcal{N}(0, P)$. Evaluating the mutual information for this distribution,

$$\begin{aligned} C_2 =& \max I(X; Y_1, Y_2) \\ =& h(Y_1, Y_2) - h(Y_1, Y_2|X) \\ =& h(Y_1, Y_2) - h(Z_1, Z_2|X) \\ =& h(Y_1, Y_2) - h(Z_1, Z_2) \end{aligned}$$

   Now since

$$(Z_1, Z_2) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}\right),$$

   we have

$$h(Z_1, Z_2) = \frac{1}{2}\log(2\pi e)^2|K_Z| = \frac{1}{2}\log(2\pi e)^2 N^2(1-\rho^2).$$

   Since $Y_1 = X + Z_1$, and $Y_2 = X + Z_2$, we have

$$(Y_1, Y_2) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} P+N & P+\rho N \\ P+\rho N & P+N \end{bmatrix}\right),$$

   and

$$h(Y_1, Y_2) = \frac{1}{2}\log(2\pi e)^2|K_Y| = \frac{1}{2}\log(2\pi e)^2(N^2(1-\rho^2) + 2PN(1-\rho)).$$

   Hence the capacity is

$$\begin{aligned} C_2 =& h(Y_1, Y_2) - h(Z_1, Z_2) \\ =& \frac{1}{2}\log\left(1 + \frac{2P}{N(1+\rho)}\right). \end{aligned}$$

   (a) $\rho = 1$. In this case, $C = \frac{1}{2}\log(1 + \frac{P}{N})$, which is the capacity of a single look channel. This is not surprising, since in this case $Y_1 = Y_2$.

   (b) $\rho = 0$. In this case,

$$C = \frac{1}{2}\log\left(1 + \frac{2P}{N}\right),$$

   which corresponds to using twice the power in a single look. The capacity is the same as the capacity of the channel $X \to (Y_1 + Y_2)$.

   (c) $\rho = -1$. In this case, $C = \infty$, which is not surprising since if we add Y_1 and Y_2, we can recover X exactly.

   Note that the capacity of the above channel in all cases is the same as the capacity of the channel $X \to Y_1 + Y_2$.

2. *Problem 4, Ch. 10, Cover&Thomas*
   By the result of Section 10.4, it follows that we will put all the signal power into the channel with less noise until the total power of noise + signal in that channel equals the noise power in the other channel. After that, we will split any additional power evenly between the two channels. Thus the combined channel begins to behave like a pair of parallel channels when the signal power is equal to the difference of the two noise powers, i.e., when $2P = \sigma_1^2 - \sigma_2^2$.