EL606: Homework 1 – Solutions

Due date: Sept. 21, 2004

1. *Problem 2, Ch. 2, Cover&Thomas*
   Consider $H(X,Y)$. We have

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y) \tag{1}$$

   Now, for $Y = g(X)$, there is no uncertainty about Y when the value of X is known. Therefore, $H(Y|X)$, and (1) implies

$$H(X) = H(Y) + H(X|Y) \geq H(Y).$$

   (a) For $Y = 2^X$, we have $H(X|Y) = 0$, and thus $H(X) = H(Y)$, since there is no uncertainty about X when the value of Y is known (g is an invertible function).

   (b) For $Y = \cos(X)$, all that we can say is that $H(X) \geq H(Y)$, with equality if cosine is one-to-one on the range of X.

2. *Problem 5, Ch. 2, Cover&Thomas*
   See the solution to the previous problem.

3. *Problem 6, Ch. 2, Cover&Thomas*
   Assume that there exists an x, say x_0 and two (or more) different values of y, say y_1 and y_2 such that $p(x_0, y_1) > 0$ and $p(x_0, y_2) > 0$. Then $0 < p(y_1|x_0) < 1$ and $0 < p(y_2|x_0) < 1$. Consequently,

$$\begin{aligned} H(Y|X) &= -\sum_x p(x) \sum_y p(y|x) \log p(y|x) \\ &\geq p(x_0)(-p(y_1|x_0) \log p(y_1|x_0) - p(y_2|x_0) \log p(y_2|x_0)) > 0. \end{aligned}$$

   Note that the conditional entropy $H(Y|X)$ is 0 if and only if Y is a function of X.

4. *Problem 16, Ch. 2, Cover&Thomas*
   Since, $p(x,y)$ is given, we can compute the marginal distributions. We have $p_X = \{2/3, 1/3\}$ and $p_Y = \{1/3, 2/3\}$. Therefore

   (a) $H(X) = \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 = 0.918$ bits $= H(Y)$.

   (b) $H(X|Y) = \frac{1}{3}H(X|Y=0) + \frac{2}{3}H(X|Y=1) = 0.667$ bits $= H(Y|X)$.

   (c) $H(X,Y) = 3 \times \frac{1}{3}\log 3 = 1.585$ bits.

   (d) $H(Y) - H(Y|X) = 0.251$ bits.

   (e) $I(X;Y) = H(Y) - H(Y|X) = 0.251$ bits.

   (f) See Figure 2.2 in Cover&Thomas.

5. *Problem 18, Ch. 2, Cover&Thomas*

   (a) $Z = X + Y$. Hence $p(Z = z|X = x) = p(Y = z - x|X = x)$.

$$\begin{aligned} H(Z|X) &= \sum p(x) H(Z|X = x) \\ &= -\sum_x p(x) \sum_z p(Z = z|X = x) \log p(Z = z|X = x) \\ &= \sum_x p(x) \sum_y p(Y = z - x|X = x) \log p(Y = z - x|X = x) \\ &= \sum p(x) H(Y|X = x) \\ &= H(Y|X). \end{aligned}$$

   If X and Y are independent, then $H(Y|X) = H(Y)$. Since $H(Z) \geq H(Z|X)$, and we have shown that $H(Z|X) = H(Y|X)$, we have $H(Z) \geq H(Y)$. Similarly, we can show that $H(Z) \geq H(X)$.

   (b) Consider the following joint distribution for X and Y Let

$$X = -Y = \begin{cases} 1 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/2 \end{cases}$$

   Then $H(X) = H(Y) = 1$, but $Z = 0$ with prob. 1 and hence $H(Z) = 0$.

   (c) Consider

$$H(X, Y, Z) = H(Z) + H(X, Y|Z) = H(X) + H(Y|X) + \underbrace{H(Z|X, Y)}_{=0}.$$

   It follows that

$$H(Z) = H(X) + \underbrace{H(Y|X)}_{\leq H(Y)} - \underbrace{H(X, Y|Z)}_{\geq 0} \leq H(X) + H(Y).$$

   We have equality iff (X, Y) is a function of Z (and thus $H(X, Y|Z) = 0$) and X and Y are independent (and thus $H(Y) = H(Y|X)$).

6. *Problem 29, Ch. 2, Cover&Thomas*

   (a) Using the chain rule for conditional entropy,

$$H(X, Y|Z) = H(X|Z) + \underbrace{H(Y|X, Z)}_{\geq 0} \geq H(X|Z),$$

   with equality iff $H(Y|X, Z) = 0$, that is, when Y is a function of X and Z. Recall the previous problem.

   (b) Using the chain rule for mutual information,

$$I(X, Y; Z) = I(X; Z) + \underbrace{I(Y; Z|X)}_{\geq 0} \geq I(X; Z),$$

   with equality iff $I(Y; Z|X) = 0$, that is, when Y and Z are conditionally independent given X.

(c) Using first the chain rule for entropy and then the definition of conditional mutual information,

$$\begin{aligned} H(X,Y,Z) - H(X,Y) = H(Z|X,Y) &= H(Z|X) - \underbrace{I(Y;Z|X)}_{\geq 0} \\ &\leq H(Z|X) = H(X,Z) - H(X), \end{aligned}$$

with equality iff $I(Y;Z|X) = 0$, that is, when Y and Z are conditionally independent given X. This inequality is known as the *strong subadditivity* of the entropy. What can you tell about it for the Markov chain $Y \to X \to Z$?

(d) Using the chain rule for mutual information,

$$I(X;Z|Y) + I(Z;Y) = I(X,Y;Z) = I(Z;Y|X) + I(X;Z)\,,$$

and therefore

$$I(X;Z|Y) = I(Z;Y|X) - I(Z;Y) + I(X;Z)\,.$$

We see that this inequality is actually an equality in all cases.