EE606: Lecture 2

Supplemental Material

Sept. 21, 2004

Proposition 1 Markov's inequality
If X is a non-negative random variable, then for any $\delta > 0$,

$$\Pr\{X \geq \delta\} \leq \frac{EX}{\delta}.$$

Proof of Proposition 1 *Let I be the random variable defined as follows:*

$$I = \begin{cases} 1 & \text{if } X \geq \delta, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$I \leq \frac{X}{\delta} \text{ and thus } EI \leq \frac{EX}{\delta}.$$

Noting that $EI = \Pr\{X \geq \delta\}$ proves the claim of the proposition. ■
A straightforward proof goes as follows. Let $p(x)$ be the PDF of X.

$$\begin{aligned} EX &= \int_0^\infty xp(x)dx \\ &= \int_0^\delta xp(x)dx + \int_\delta^\infty xp(x)dx \\ &\geq \int_\delta^\infty xp(x)dx \\ &\geq \int_\delta^\infty \delta p(x)dx \\ &= \delta \Pr\{X \geq \delta\}. \end{aligned}$$

Rearranging sides and dividing by δ we get,

$$\Pr\{X \geq \delta\} \leq \frac{EX}{\delta}.$$ ■

Given δ, the distribution achieving

$$\Pr\{X \geq \delta\} = \frac{EX}{\delta},$$

is

$$X = \begin{cases} \delta & \text{with probability } \mu/\delta, \\ 0 & \text{with probability } 1 - \mu/\delta, \end{cases}$$

where $\mu \leq \delta$.

Proposition 2 Chebyshev's inequality.
Let Y be a random variable with mean μ and variance σ^2. By letting $X = (Y-\mu)^2$, show that for any $\epsilon > 0$,

$$\Pr\{|Y-\mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}.$$

Proof of Proposition 2 *Letting $X = (Y-\mu)^2$ in Markov's inequality,*

$$\begin{aligned} \Pr\{(Y-\mu)^2 > \epsilon^2\} &\leq \Pr\{(Y-\mu)^2 \geq \epsilon^2\} \\ &\leq \frac{E(Y-\mu)^2}{\epsilon^2} \\ &= \frac{\sigma^2}{\epsilon^2}, \end{aligned}$$

and noticing that $\Pr\{(Y-\mu)^2 > \epsilon^2\} = \Pr\{|Y-\mu| > \epsilon\}$, we get,

$$\Pr\{|Y-\mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}.$$

Thheorem 1 The weak law of large numbers.
Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Let $\overline{Z}_n = \frac{1}{n}\sum_{i=1}^n Z_i$ be the sample mean. Then, for any $\epsilon > 0$,

$$\Pr\left\{\left|\overline{Z}_n - \mu\right| > \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}.$$

Thus $\Pr\left\{\left|\overline{Z}_n - \mu\right| > \epsilon\right\} \to 0$ as $n \to \infty$.

Proof of Theorem 1 *Letting Y in Chebyshev's inequality from part equal $\bar{Z}_n$, and noticing that $E\bar{Z}_n = \mu$ and $Var(\bar{Z}_n) = \frac{\sigma^2}{n}$ (i.e., $\bar{Z}_n$ is the sum of n iid r.v.'s, $\frac{Z_i}{n}$, each with variance $\frac{\sigma^2}{n^2}$), we have,*

$$\Pr\{|\overline{Z}_n - \mu| > \epsilon\} \leq \frac{\sigma^2}{n\epsilon^2}.$$

This result is known as the weak law of large numbers, and it states that the sample mean converges to the mean of the distribution *in probability*. We will also write

$$\overline{Z}_n = \frac{1}{n}\sum_{i=1}^n Z_i \to \mu \qquad \text{in probability.}$$