Lecture 1, E6713

COMMUNICATION NETWORK – A Mathematical Model

- A network is represented as a directed, acyclic graph $G = (V, E)$.
- G has unit-capacity edges and parallel edges are allowed.
- There are h unit-rate, co-located, information sources $S_1, \ldots, S_h$.
- There are N receivers $R_1, \ldots, R_N$ located at N distinct nodes.
- Can all sources *simultaneously* transmit information to all receivers?
  (We are interested in the transmission at full rate.)

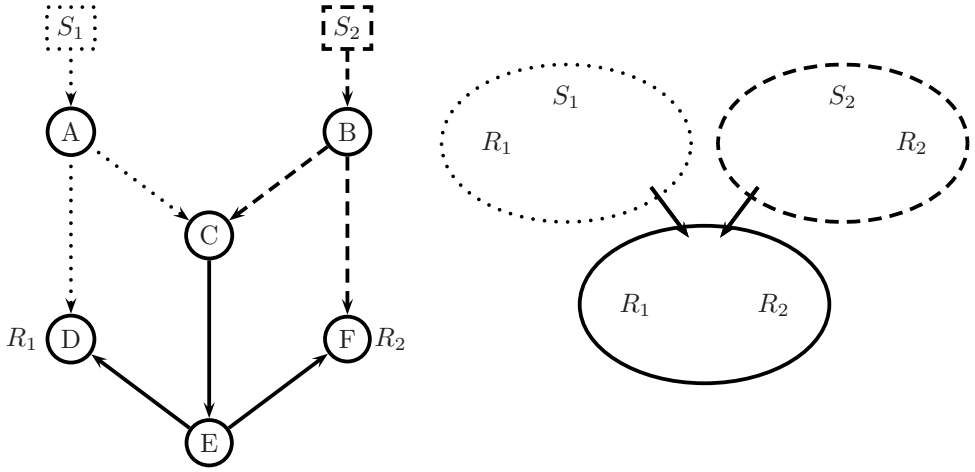
NETWORK MULTICAST – Throughput

- Can all sources simultaneously transmit information to receiver j?
  Yes, if between the sources and the j-th receiver node
  - the number of edges in the min-cut is h (or equivalently)
  - there are h edge-disjoint paths (S_i, R_j) for $1 \leq i \leq h$.

  [Ford, Fulkerson], [Elias, Feinstein, Shannon] $\sim$ 50s
- Can all sources simultaneously transmit information to all receivers?
  Yes, if in addition each node of G can re-encode information. [Alshwede, Cai, Li, Yeung] $\sim$ 2000

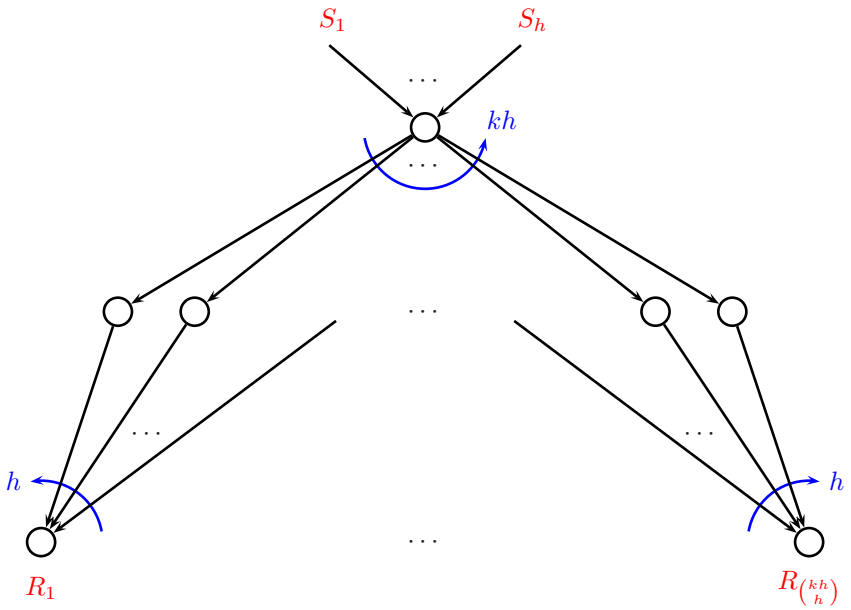
NETWORK MULTICAST – Linear Combining

- Source S_i emits σ_i which is an element of some finite field $\mathbb{F}_q$.
- Each edge carries a linear combination of its parent node inputs.
- Consequently, each edge carries a linear combination of source symbols.
- The h edges a receiver observes should carry independent linear combination of source symbols.
- Network code design problem: How should nodes combine their inputs to ensure this independence?



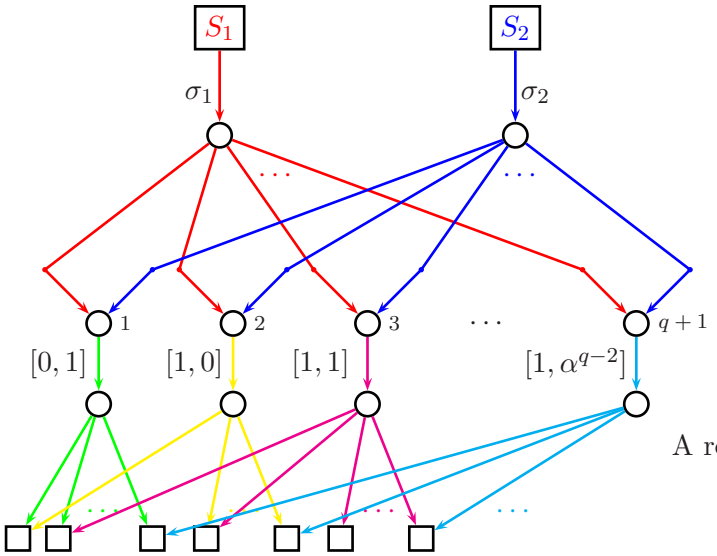
(a) (b)

Figure 1: A network with two sources and two receivers and its information flow decomposition. Each receiver observes two distinct flows.



Encode the h source symbols by an (mh, h) RS code.

Figure 2: Combination $B(h, k)$ network.



α is a primitive element of $\mathbb{F}_q$.

A receiver observes $\sigma_1 + \alpha^i \sigma_2$ and $\sigma_1 + \alpha^j \sigma_2$.

$\leftarrow \binom{q+1}{2}$ receivers.

Figure 3: A network with two sources and $\binom{q+1}{2}$ receivers.

AVERAGE THROUGHPUT WITH ROUTING

- T_j^r is the throughput to receiver j when only routing is used.
- Compare the average routing throughput with coding throughput:

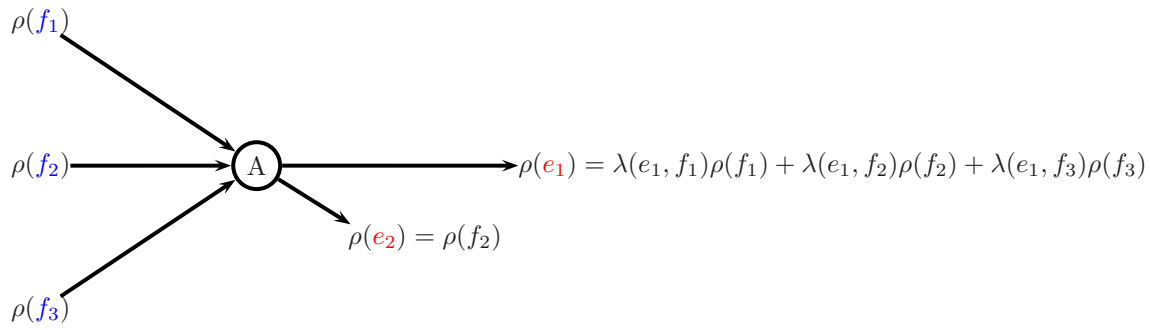
$$T^r = \frac{1}{N}\sum_{i=1}^{N} T_i^r \quad \text{vs} \quad T^c = \frac{1}{N}\sum_{i=1}^{N} T_i^c = 2.$$

- Route σ_1 through one half of the $q+1$ intermediate nodes, and σ_2 through the other half.
- The routing throughput (for even $q+1$):

$$T^r = \frac{1}{\binom{q+1}{2}}\left[\frac{q+1}{2}\left(\frac{q+1}{2}-1\right)\cdot 1 + \left(\frac{q+1}{2}\right)^2 \cdot 2\right] > \frac{3}{4}\cdot 2$$

NETWORK MULTICAST – Linear Combining

- Source S_i emits σ_i which is an element of some finite field $\mathbb{F}_q$.
- Each node of G
  - receives an element of $\mathbb{F}_q$ from each input edge, and then
  - forwards a linear combination of its inputs to each output edge.



NETWORK MULTICAST – Coding Vectors

- Each edge carries a linear combination of its parent node inputs.
- Each edge e carries a linear combination of source symbols:

$$c_1(e)\sigma_1 + \cdots + c_h(e)\sigma_h = \begin{bmatrix} c_1(e) \dots c_h(e) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_h \end{bmatrix}$$

- $[c_1(e) \dots c_h(e)] \in \mathbb{F}_q^n$ is the coding vector of edge e.

NETWORK MULTICAST – Decoding for Receiver j

- ρ_i^j is the symbol on the last edge of the path (S_i, R_j).
- c_i^j is the coding vector of the last edge on the path (S_i, R_j).
- $\mathbf{C}_j$ is the matrix whose i-th row is c_i^j.
- The receiver j has to solve the following system of linear equations:

$$\begin{bmatrix} \rho_1^j \\ \vdots \\ \rho_h^j \end{bmatrix} = \mathbf{C}_j \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_h \end{bmatrix}.$$

NETWORK MULTICAST – The Code Design Problem

Select a coding vector for each edge e of the network so that

1. the matrix $\mathbf{C}_j$ is full rank for each receiver j and
2. the coding vector of e is in the linear span of the coding vectors of the input edges to the parent node of e.

CODING VECTORS – Networks with Two Sources

- For edge e, coding vector $[c_1(e)\, c_2(e)]$ is in $\mathbb{F}_q^2$.
- Let $\mathcal{A}$ be the following set of vectors:

$$[0\,1],\ [1\,0],\ \text{and } [1\,\alpha^i] \text{ for } 0 \le i \le q-2,$$

  where α is a primitive element of $\mathbb{F}_q$.

- Consider any two different vectors in $\mathcal{A}$:
  - they are linearly independent, and
  - any vector in $\mathcal{A}$ is in their linear span.

CODING VECTORS – Vectors in General Position

- Let $\mathcal{A}$ be a set of vectors in $\mathbb{F}_q^h$.
- We require that any h vectors in $\mathcal{A}$ be linearly independent.
- What is the size $g(h,q)$ of the largest such $\mathcal{A}$?

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{bmatrix} \Rightarrow\ g(h,q) \ge h{+}1$$

VECTORS IN GENERAL POSITION – MDS Codes over $\mathbb{F}_q$

- Consider an MDS code of dimension h and generator matrix G.
- The columns of G are vectors in general position in $\mathbb{F}_q^h$.
- What is the maximum length $g(h,q)$ such code can have?

$$\mathcal{A} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{h-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{h-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_q & x_q^2 & \dots & x_q^{h-1} \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \Rightarrow\ g(h,q) \ge q{+}1$$