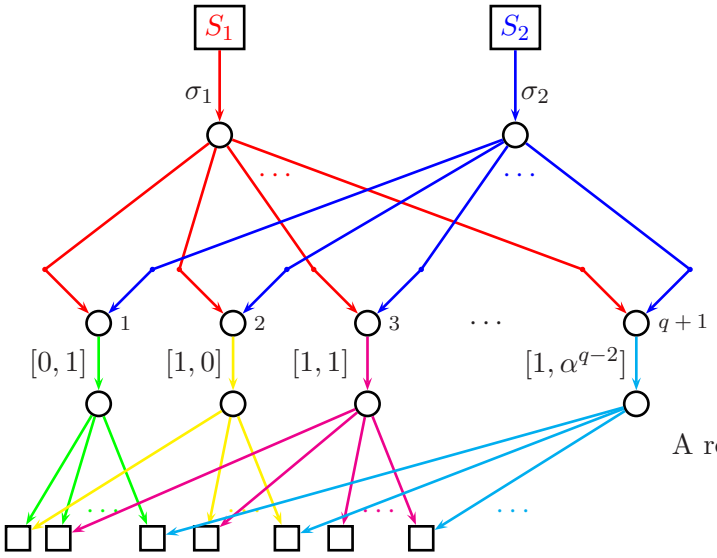
Lecture 3, E6713

NETWORK MULTICAST – An Example



α is a primitive element of $\mathbb{F}_q$.

A receiver observes $\sigma_1 + \alpha^i\sigma_2$ and $\sigma_1 + \alpha^j\sigma_2$.

$\leftarrow \binom{q+1}{2}$ receivers.

Figure 1: A network with two sources and $\binom{q+1}{2}$ receivers.

AVERAGE THROUGHPUT WITH ROUTING

- T_j^r is the throughput to receiver j when only routing is used.
- Compare the average routing throughput with coding throughput:

$$T^r = \frac{1}{N}\sum_{i=1}^{N} T_i^r \quad \text{vs} \quad T^c = \frac{1}{N}\sum_{i=1}^{N} T_i^c = 2.$$

- Route σ_1 through one half of the $q-1$ intermediate nodes, and σ_2 through the other half.
- The routing throughput (for even $q-1$):

$$T^r = \frac{1}{\binom{q-1}{2}}\left[\frac{q-1}{2}\left(\frac{q-1}{2}-1\right)\cdot 1 + \left(\frac{q-1}{2}\right)^2 \cdot 2\right] > \frac{3}{4}\cdot 2$$

CODING v.s. ROUTING – Throughput Measures

- $T_j^c = h$ is the throughput to receiver j when coding is used.
- T_j^r is the throughput to receiver j when only routing is used.
- Compare the average routing throughput with coding throughput:

$$T^r = \frac{1}{N}\sum_{i=1}^{N} T_i^r \quad \text{vs} \quad T^c = \frac{1}{N}\sum_{i=1}^{N} T_i^c = h.$$

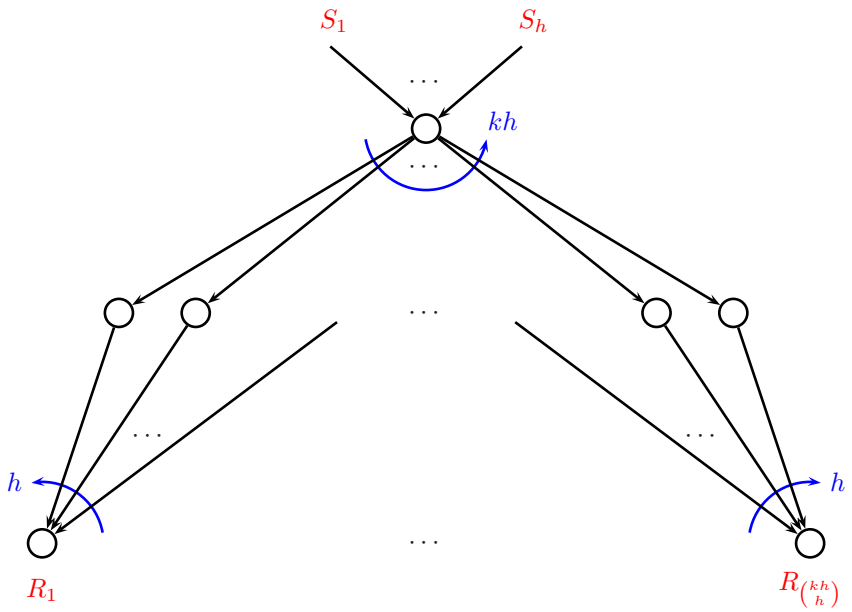
  (We are interested in the maximal routing throughput.)

- There are other throughput measures, and coding benefits depend on the measure.

CODING v.s. ROUTING – Measures and Scenarios

- Individual receiver requirements: symmetric or sum.
- Type of routing: integral or fractional.
- Communications schemes: time sharing or channel coding.
- Type of network: directed or undirected. General results for symmetric throughput and fractional routing:
  1. In undirected networks, coding can at most double the throughput. (Li&Li '04)
  2. In both undirected and directed networks, coding benefits are equal to the integrality gap of the standard Steiner tree problem. (Agrawal&Charikar '04).

NETWORK MULTICAST – An Example



Encode the h source symbols by an (mh, h) RS code.

Figure 2: Combination $B(h, k)$ network.

AVERAGE THROUGHPUT WITH ROUTING – An Occupancy Model

- There are h urns (sources) and each receiver has h balls.
- T_i representing the number of observed sources (filled urns) is given by

$$\Pr\{T_i = k\} = \binom{h}{k}\left(1 - \frac{h-k}{h}\right)^h \Pr\{\mu_0(k) = 0\}$$

$$\text{where } \Pr\{\mu_0(k) = 0\} = \sum_{l=0}^{k}\binom{k}{l}(-1)^l\left(1 - \frac{l}{k}\right)^h.$$

- A receiver observes

$$h[1 - (1 - 1/h)^h] \gtrsim h(1 - e^{-1}) > h/2$$

sources on the average.

SYMMETRIC THROUGHPUT – A Coding Theorem

- The probability that a receiver does not observe source S_i is

$$\epsilon = (1 - 1/h)^h.$$

- The expected value of the throughput is

$$h[1 - (1 - 1/h)^h] = h(1 - \epsilon).$$

- Theorem: There exist a sequence of channel codes of rates

$$k/n \to 1 - \epsilon$$

and a routing strategy such that the symmetric throughput

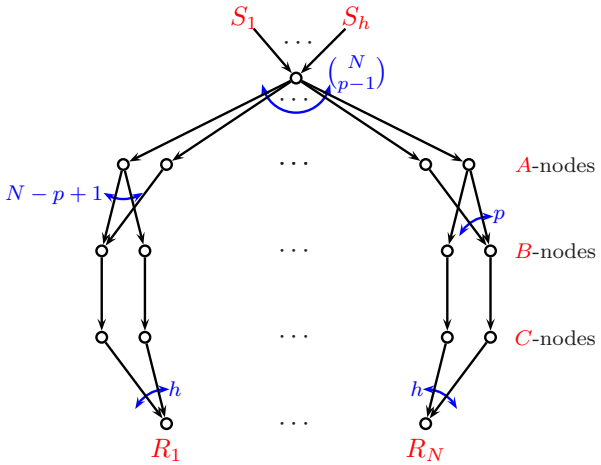
$$T_i(n) \to hk/n \to h(1 - \epsilon) \text{ as } n \to \infty.$$

AVERAGE THROUGHPUT BENEFITS OF NETWORK CODING

- In our two-source network example, $T^r \geq 3T_c/4$.
- For general networks with two sources, $T^r \geq T_c/2 + 1/N$.
- For networks with two receivers, $T^r \geq T_c/2$.
- For several other classes of networks, $T^r \geq T_c/2$.
- There are networks for which

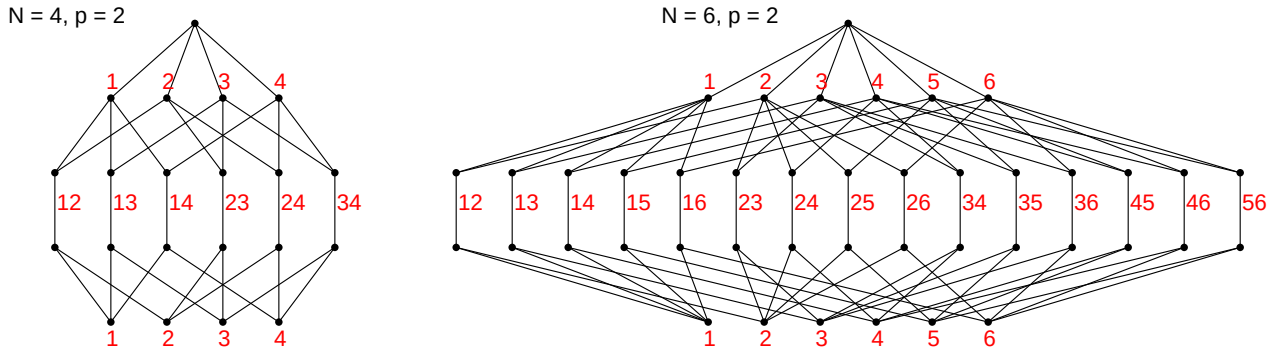
$$T^r \simeq \frac{1}{\sqrt{N}} \cdot T^c.$$

AVERAGE THROUGHPUT WITH ROUTING



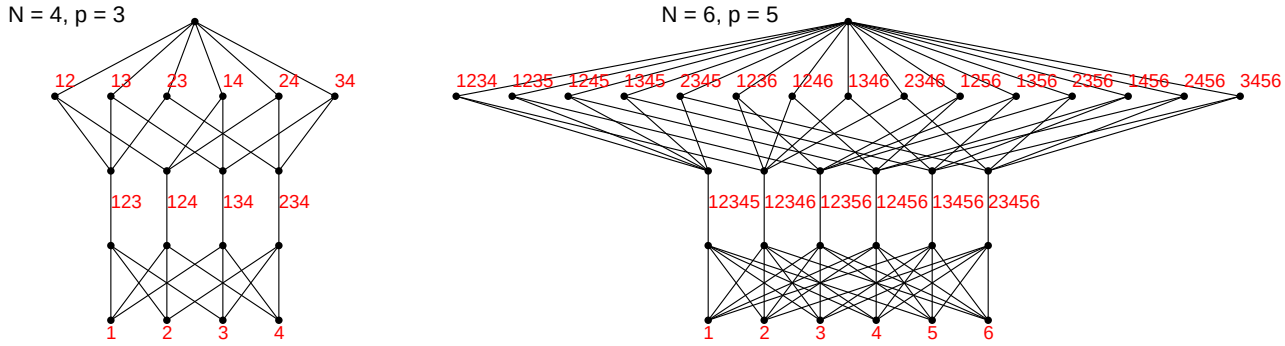
We have $\frac{T^r}{T^c} = \frac{p-1}{N-p+1} + \frac{1}{p}$. For $p = \frac{N+1}{\sqrt{N}+1}$, we get $\frac{T^r}{T^c} = 2\,\frac{\sqrt{N}}{1+N} \lesssim \frac{2}{\sqrt{N}}$.

A NETWORK CODE FOR THE EXAMPLE – $p = 2$



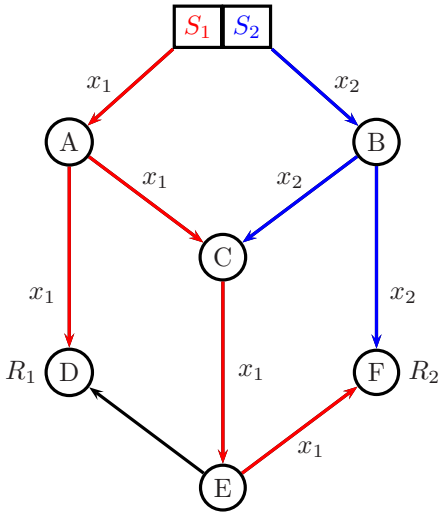
- The number of information sources is $h = N - 1$.
- Remove the A-node with label N. B-nodes add their inputs.
- Receiver i gets σ_i and $\sigma_j + \sigma_i$ for $i, j = 1, \ldots, N-1$ and $j \neq i$.

A NETWORK CODE FOR THE EXAMPLE – $p = N - 1$



- The number of information sources is $h = N - 1$.
- Remove the A-node whose label contains N.
- The $B - C$ edges carry the following: $\sum_j \sigma_j$, σ_i for $i = 1, \ldots, N - 1$.

ROUTING THROUGHPUT – Packing Steiner Trees



STEINER TREE PROBLEMS – Graph Instance $\{G, S, \mathcal{R}, w, c\}$

minimum weight:

$$\min \sum_{e \in E} w_e x_e$$

$$\sum_{e \in \delta(\mathcal{D})} x_e \geq 1, \quad \forall\ \mathcal{D}\text{: } \mathcal{D} \text{ is separating}$$

$$x_e \in \{0,1\}, \quad \forall\ e \in E$$

packing:

$$\max \sum_{t \in \tau} y_t$$

$$\sum_{t \in \tau : e \in t} y_t \leq c_e, \quad \forall\ e \in E$$

$$y_t \geq 0, \quad \forall\ t \in \tau$$

packing dual:

$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in t} x_e \geq 1, \quad \forall\ t \in \tau$$

$$x_e \geq 0, \quad \forall\ e \in E$$

CODING ALPHABET SIZE

Theorem (Fragouli&Soljanin '04): For networks with two sources and N receivers, the code alphabet $\mathbb{F}_q$ of size

$$q = \lfloor \sqrt{2N - 7/4} + 1/2 \rfloor$$

is sufficient. There exist configurations for which it is necessary.

NETWORK CODES FOR THE EXAMPLE – Alphabet Size

- The code design generalizes to arbitrary p.
- The binary alphabet is sufficient.
- What about coding randomly over the alphabet $\mathbb{F}_q$?
- A bound on the probability that all receivers can decode

$$P_N^d \geq \left(1 - \frac{N}{q}\right)^{\binom{N}{p}} \approxeq e^{-\frac{N\binom{N}{p}}{q}}.$$