Capacity Outer Bounds for Broadcast Channels

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***Abstract*— Outer bounds on the capacity region of broadcast channels are reviewed and a new outer bound is presented.**

I. Introduction

The two-receiver discrete memoryless broadcast channel (BC) is depicted in Fig. 1 (see [1], [2]). Three sources put out three statistically independent messages W_0, W_1, W_2 with entropies $H(W_k) = nR_k$, $k = 0, 1, 2$, where n is the number of times we use the channel. The message W_0 is a *common* message that is destined for both receivers, while W_1 and W_2 are destined for receivers 1 and 2, respectively.

The transmitter maps (encodes) the three messages to a string $X^n = X_1, X_2, \ldots, X_n$, where the X_i, $i = 1, 2, \ldots, n$, are random variables with alphabet (or range) $\mathcal{X}$. The channel puts out the string of pairs

$$(Y^n, Z^n) = ((Y_1, Z_1), (Y_2, Z_2), \ldots, (Y_n, Z_n))$$

where (Y_i, Z_i) has alphabet $\mathcal{Y} \times \mathcal{Z}$ and is a noisy function of X_i for all i. We write the channel probability distribution as $P_{YZ|X}(\cdot)$. Receiver 1 maps (decodes) its output string Y^n to a pair $(\hat{W}_0(1), \hat{W}_1)$, and similarly for receiver 2. The error probability is

$$P_e = \Pr\left[(\hat{W}_0(1), \hat{W}_0(2), \hat{W}_1, \hat{W}_2) \neq (W_0, W_0, W_1, W_2)\right] \quad (1)$$

and a rate triple (R_0, R_1, R_2) is said to be *achievable* if, for any $\epsilon > 0$, we can find an encoder and two decoders such that $P_e < \epsilon$ for some n. The closure of the set of achievable rate triples is the *capacity region* $\mathcal{C}$.

The capacity regions of some classes of BCs have been characterized, including degraded BCs [3]–[5], deterministic BCs [6], [7], semi-deterministic BCs [8], BCs with degraded message sets [9], more capable BCs [10], the sum and product of two reversely degraded BCs [11], and Gaussian multiple-input multiple-output (MIMO) BCs with $R_0 = 0$ [12]. However, the capacity region of the general BC is still unknown. Inner bounds on the capacity region are developed in [1], [13]–[17], and outer bounds on the capacity region appear in [8], [15], [17]–[19]. The focus of this paper is on developing a new outer bound that generalizes the recent results of [17]–[19].

II. Existing Outer Bounds

Körner and Marton derived the following outer bound [15].

***Lemma** 1:* In the plane $R_0 = 0$, $\mathcal{C}$ is a subset of the region

$$\bar{\mathcal{R}}_{KM} = \bigcup_{p(u,v)p(x|u,v)p(y,z|x)} \left\{ \begin{array}{l} (R_1, R_2) : R_1 \geq 0, R_2 \geq 0, \\ R_1 \leq I(U;Y), \\ R_2 \leq I(V;Z), \\ R_1 + R_2 \leq I(U;Y) + I(V;Z|U), \\ R_1 + R_2 \leq I(V;Z) + I(U;Y|V) \end{array} \right\}. \quad (2)$$

***Remark** 1:* One can restrict X to be a deterministic function of (U, V) [19].

Recently, two new bounds on the capacity appeared in [17], [18] and [19]. The first bound is due to Liang and Kramer.

***Lemma** 2:* $\mathcal{C}$ is a subset of the region

$$\bar{\mathcal{R}}^0_{LK} = \bigcup_{p(t,u,v,x)p(y,z|x)} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\left\{I(T;Y), I(T;Z)\right\}, \\ R_0 + R_1 \leq I(X;Y), \\ R_0 + R_2 \leq I(T,U;Z), \\ R_0 + R_1 + R_2 \leq I(T;Y) + I(X;Y|T,U) \\ \qquad\qquad + I(U;Z|T) \\ R_0 + R_1 + R_2 \leq I(T,U;Z) + I(X;Y|T,U), \\ R_0 + R_1 \leq I(T,V;Y), \\ R_0 + R_2 \leq I(X;Z), \\ R_0 + R_1 + R_2 \leq I(T,V;Y) + I(X;Z|T,V), \\ R_0 + R_1 + R_2 \leq I(T;Z) + I(V;Y|T) \\ \qquad\qquad + I(X;Z|T,V) \end{array} \right\} \quad (3)$$

***Remark** 2:* Let $\bar{\mathcal{R}}_{LK}$ be the set of rate pairs (R_1, R_2) in $\bar{\mathcal{R}}^0_{LK}$ when $R_0 = 0$. We identify $U' = (T, U)$ and $V' = (T, V)$ and find that $\bar{\mathcal{R}}_{LK} \subseteq \bar{\mathcal{R}}_{KM}$.

The second recent outer bound is due to Nair and El Gamal.

***Lemma** 3:* $\mathcal{C}$ is a subset of the region

$$\bar{\mathcal{R}}^0_{NEG} = \bigcup_{p(u)p(v)p(w|u,v)p(x|u,v,w)p(y,z|x)} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\{I(W;Y), I(W;Z)\}, \\ R_0 + R_1 \leq I(UW;Y), \\ R_0 + R_2 \leq I(VW;Z), \\ R_0 + R_1 + R_2 \leq I(UW;Y) + I(V;Z|UW), \\ R_0 + R_1 + R_2 \leq I(VW;Z) + I(U;Y|VW) \end{array} \right\}. \quad (4)$$

***Remark** 3:* Let $\bar{\mathcal{R}}_{NEG}$ be the set of rate pairs (R_1, R_2) in $\bar{\mathcal{R}}^0_{NEG}$ when $R_0 = 0$. We identify $U' = (W, U)$ and $V' =$

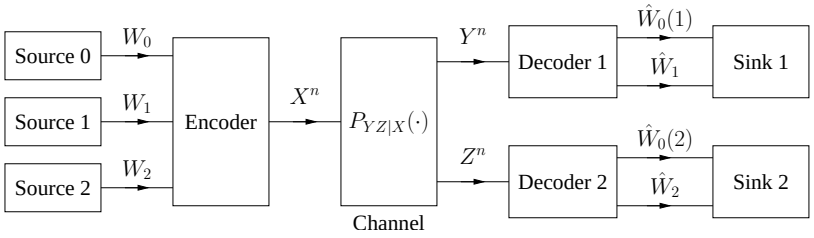


Fig. 1. The two-receiver broadcast channel.

(W, V) and find that $\bar{\mathcal{R}}_{NEG} \subseteq \bar{\mathcal{R}}_{KM}$. In fact, it turns out that $\bar{\mathcal{R}}_{NEG} = \bar{\mathcal{R}}_{KM}$ [20].

***Remark** 4:* It is not clear if one of the sets $\bar{\mathcal{R}}_{LK}$ or $\bar{\mathcal{R}}_{NEG}$ includes the other, or neither includes the other.

***Remark** 5:* $\bar{\mathcal{R}}^0_{LK} = \bar{\mathcal{R}}^0_{NEG} = \mathcal{C}$ for deterministic BCs, namely

$$\mathcal{C} = \bigcup_{p(q)p(x|q)p(y,z|x)} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\{I(Q;Y), I(Q;Z)\} \\ R_0 + R_1 \leq H(Y) \\ R_0 + R_2 \leq H(Z) \\ R_0 + R_1 + R_2 \leq I(Q;Y) + H(YZ|Q) \\ R_0 + R_1 + R_2 \leq I(Q;Z) + H(YZ|Q) \end{array} \right\}. \quad (5)$$

***Remark** 6:* $\bar{\mathcal{R}}_{LK} = \bar{\mathcal{R}}_{NEG} = \mathcal{C}$ for semi-deterministic BCs without a common message, namely

$$\mathcal{C} = \bigcup_{p(w_2)p(x|w_2)p(y,z|x)} \left\{ \begin{array}{l} (R_1, R_2) : R_1 \geq 0, R_2 \geq 0, \\ R_1 \leq H(Y), \\ R_2 \leq I(W_2;Z), \\ R_1 + R_2 \leq I(Y|W_2) + H(W_2;Z) \end{array} \right\}. \quad (6)$$

***Remark** 7:* $\bar{\mathcal{R}}^0_{LK} = \mathcal{C}$ for semi-deterministic BCs with a common message [8], namely

$$\mathcal{C} = \bigcup_{p(q,w_2)p(x|q,w_2)p(y,z|x)} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\{I(Q;Y), I(Q;Z)\}, \\ R_0 + R_1 \leq H(Y), \\ R_0 + R_2 \leq I(QW_2;Z), \\ R_0 + R_1 + R_2 \leq H(Y|QW_2) + I(QW_2;Z), \\ R_0 + R_1 + R_2 \leq I(Q;Y) + H(Y|QW_2) \\ \qquad\qquad + I(W_2;Z|Q) \end{array} \right\}. \quad (7)$$

The last bound in (7) can be derived from the first bound on $R_0 + R_1 + R_2$ in (3). However, it is not clear if the last bound in (7) can be derived from $\bar{\mathcal{R}}^0_{NEG}$.

III. New-Jersey Outer Bound

We present a new outer bound, called the *New-Jersey Outer Bound*, that is based on ideas of [8], [17]–[19].

***Theorem** 1:* $\mathcal{C}$ is a subset of the New-Jersey region

$$\bar{\mathcal{R}}^0_{NJ1} = \bigcup_{\substack{p(w_0)p(w_1)p(w_2)p(u,v|w_0,w_1,w_2) \\ p(x|u,v,w_0,w_1,w_2)p(y,z|x)}} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\{I(W_0;Y|U), I(W_0;Z|V)\} \\ R_1 \leq I(W_1;Y|U) \\ R_2 \leq I(W_2;Z|V) \\ R_0 + R_1 \leq \min\{I(W_0W_1;Y|U), \\ \qquad I(W_1;Y|W_0UV) + I(W_0U;Z|V)\} \\ R_0 + R_2 \leq \min\{I(W_0W_2;Z|V), \\ \qquad I(W_0V;Y|U) + I(W_2;Z|W_0UV)\} \\ R_0 + R_1 + R_2 \leq I(W_1;Y|W_0W_2UV) \\ \qquad + I(W_0W_2U;Z|V) \\ R_0 + R_1 + R_2 \leq I(W_2;Z|W_0W_1UV) \\ \qquad + I(W_0W_1V;Y|U) \\ R_0 + R_1 + R_2 \leq I(W_0UV;Y) \\ \qquad + I(W_1;Y|W_0W_2UV) + I(W_2;Z|W_0UV) \\ R_0 + R_1 + R_2 \leq I(W_0UV;Z) \\ \qquad + I(W_2;Z|W_0W_1UV) + I(W_1;Y|W_0UV) \end{array} \right\} \quad (8)$$

where X is a deterministic function of (W_0, W_1, W_2, U, V), and W_0, W_1, W_2 are uniformly distributed.

***Remark** 8:* $\bar{\mathcal{R}}^0_{NJ1}$ is a subset of both $\bar{\mathcal{R}}^0_{LK}$ and $\bar{\mathcal{R}}^0_{NEG}$.

We present several weakened versions of $\bar{\mathcal{R}}^0_{NJ1}$ that are useful for comparing regions.

***Theorem** 2:* $\bar{\mathcal{R}}^0_{NJ1}$ is a subset of the region

$$\bar{\mathcal{R}}^0_{NJ2} = \bigcup_{\substack{p(w_0)p(w_1)p(w_2)p(u,v|w_0,w_1,w_2) \\ p(x|u,v,w_0,w_1,w_2)p(y,z|x)}} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\{I(W_0U;Y), I(W_0V;Z)\} \\ R_0 + R_1 \leq I(W_0W_1U;Y) \\ R_0 + R_2 \leq I(W_0W_2V;Z) \\ R_0 + R_1 + R_2 \leq I(W_1;Y|W_0W_2UV) \\ \qquad + I(W_0W_2UV;Z) \\ R_0 + R_1 + R_2 \leq I(W_2;Z|W_0W_1UV) \\ \qquad + I(W_0W_1VU;Y) \\ R_0 + R_1 + R_2 \leq I(W_0UV;Y) \\ \qquad + I(W_1;Y|W_0W_2UV) + I(W_2;Z|W_0UV) \\ R_0 + R_1 + R_2 \leq I(W_0UV;Z) \\ \qquad + I(W_2;Z|W_0W_1UV) + I(W_1;Y|W_0UV) \end{array} \right\} \quad (9)$$

where X is a deterministic function of (W_0, W_1, W_2, U, V).

***Theorem** 3:* $\bar{\mathcal{R}}^0_{NJ2}$ is a subset of the region

$$\bar{\mathcal{R}}^0_{NJ3} = \bigcup_{p(w_1)p(w_2)p(q|w_1,w_2)p(x|w_1,w_2,q)p(y,z|x)} \left\{ \begin{array}{l} (R_0, R_1, R_2) : R_0 \geq 0, R_1 \geq 0, R_2 \geq 0, \\ R_0 \leq \min\{I(Q;Y), I(Q;Z)\} \\ R_0 + R_1 \leq I(QW_1;Y) \\ R_0 + R_2 \leq I(QW_2;Z) \\ R_0 + R_1 + R_2 \leq I(W_1;Y|QW_2) + I(QW_2;Z) \\ R_0 + R_1 + R_2 \leq I(W_2;Z|QW_1) + I(QW_1;Y) \\ R_0 + R_1 + R_2 \leq I(Q;Y) + I(W_1;Y|QW_2) \\ \qquad\qquad +I(W_2;Z|Q) \\ R_0 + R_1 + R_2 \leq I(Q;Z) + I(W_2;Z|QW_1) \\ \qquad\qquad +I(W_1;Y|Q) \end{array} \right\} \tag{10}$$

where X is a deterministic function of (W_1, W_2, Q).

Proof: Define $Q := (W_0, U, V)$ and apply to $\bar{\mathcal{R}}^0_{NJ2}$. ■

***Corollary** 1:* For $R_0 = 0$, the region $\bar{\mathcal{R}}^0_{NJ3}$ reduces to

$$\bar{\mathcal{R}}_{NJ3} = \bigcup_{p(w_1)p(w_2)p(q|w_1,w_2)p(x|w_1,w_2,q)p(y,z|x)} \left\{ \begin{array}{l} (R_1, R_2) : R_1 \geq 0, R_2 \geq 0, \\ R_1 \leq I(QW_1;Y) \\ R_2 \leq I(QW_2;Z) \\ R_1 + R_2 \leq I(W_1;Y|QW_2) + I(QW_2;Z) \\ R_1 + R_2 \leq I(W_2;Z|QW_1) + I(QW_1;Y) \\ R_1 + R_2 \leq I(Q;Y) + I(W_1;Y|QW_2) + I(W_2;Z|Q) \\ R_1 + R_2 \leq I(Q;Z) + I(W_2;Z|QW_1) + I(W_1;Y|Q) \end{array} \right\} \tag{11}$$

where X is a deterministic function of (W_1, W_2, Q).

***Remark** 9:* Comparing the bounds (8), (9), (10) and (11) with the bounds (4), we find that

$$\begin{array}{l} \bar{\mathcal{R}}^0_{NJi} \subseteq \bar{\mathcal{R}}^0_{NEG}, \quad \text{for } i = 1, 2, 3; \\ \bar{\mathcal{R}}_{NJ3} \subseteq \bar{\mathcal{R}}_{NEG} \end{array} \tag{12}$$

Thus, by Remarks 7 and 8, we have $\bar{\mathcal{R}}^0_{NJi} = \mathcal{C}$ for semi-deterministic BCs and $i = 1, 2, 3$.

***Remark** 10:* The bounds $\bar{\mathcal{R}}^0_{NJi}$ for $i = 1, 2$ given in (8) and (9) remain the same if we restrict X to be a deterministic function of (W_0, W_1, W_2, U, V). The bounds $\bar{\mathcal{R}}^0_{NJ3}$ and $\bar{\mathcal{R}}_{NJ3}$ given in (10) and (11) remain the same if we restrict X to be a deterministic function of (Q, W_1, W_2).

IV. Conclusions

A new outer bound on the capacity region of two-receiver BCs was presented. However, it is not yet clear if this bound improves on other recent bounds.

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