Feedback Strategies for a Class of Two-User Multiple-Access Channels

Gerhard Kramer, *Member, IEEE*

***Abstract*—Feedback strategies are presented for the class of two-user discrete memoryless multiple-access channels for which one of the inputs to the channel is determined by the second input and the channel output. The strategies achieve any rate point inside the capacity region of these channels.**

***Index Terms*—Capacity, feedback, multiple-access channel, strategy.**

I. Introduction

There is as at present no single-letter characterization of the capacity region of the general two-user discrete memoryless multiple-access channel with feedback (MAC-FB). However, for the class of channels where one of the channel inputs, say X_1, is determined by the other input X_2 and the channel output Y, i.e., $H(X_1|X_2Y)=0$, the feedback capacity region was shown by Willems [1] to be the rate region $\mathcal{R}^{\mathrm{CL}}$ obtained by Cover and Leung [2]. This rate region is the set of pairs (R_1, R_2) for which

$$0 \le R_1 \le I(X_1; Y|X_2V) \tag{1}$$

$$0 \le R_2 \le I(X_2; Y|X_1V) \tag{2}$$

$$0 \le R_1 + R_2 \le I(X_1X_2; Y) \tag{3}$$

where V is any discrete random variable with cardinality $|\mathcal{V}|$ at most $\min(|\mathcal{X}_1| \cdot |\mathcal{X}_2|+1, |\mathcal{Y}|+2)$, and the probability distribution $p_{VX_1X_2Y}$ of the random variables V, X_1, X_2, and Y factors as

$$p_{VX_1X_2Y} = p_V \cdot p_{X_1|V} \cdot p_{X_2|V} \cdot p_{Y|X_1X_2}. \tag{4}$$

For example, the binary adder channel with feedback (BAC-FB) falls into this class of channels. For the BAC-FB, one need consider $|\mathcal{V}| \le 5$ only and, if one chooses $\Pr(V=0)=\Pr(V=1)=1/2$ and $\Pr(X_1 \ne V)=\Pr(X_2 \ne V)=0.23766$, one finds that the rate pair $(0.7911, 0.7911)$ can be approached [3] and is the equal-rate point on the boundary of the capacity region [4].

The rate region $\mathcal{R}^{\mathrm{CL}}$ was derived in [2] using *block Markov superposition encoding*, a technique which relies on the use of randomly chosen block codes. Alternatively, one can consider constructive strategies which exploit feedback strategies for single-user channels. This approach was taken by Vinck [5]–[7], who found that for the BAC-FB the rate pair $(0.7909, 0.7909)$ can be approached by using a repetition strategy of Schalkwijk [8]. In fact, we show that by choosing a better strategy one can approach the rate point $(0.7911, 0.7911)$.

We generalize the strategies of [5]–[7]. The structure of our strategies is based on the superposition coding structure of Cover and Leung [2], except that now the (single-user) *random codes* have been replaced by (single-user) feedback *strategies*. A related approach was developed by Ooi in [9, Ch. 5]. However, his strategies achieve only those points inside the *feedback-free* capacity region. Ooi's strategies are based on work by Ahlswede [10] and are extensions of a strategy described by Weldon [11].

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The author is with Endora Tech AG, CH-4051 Basel, Switzerland (e-mail: gkramer@endora.ch).

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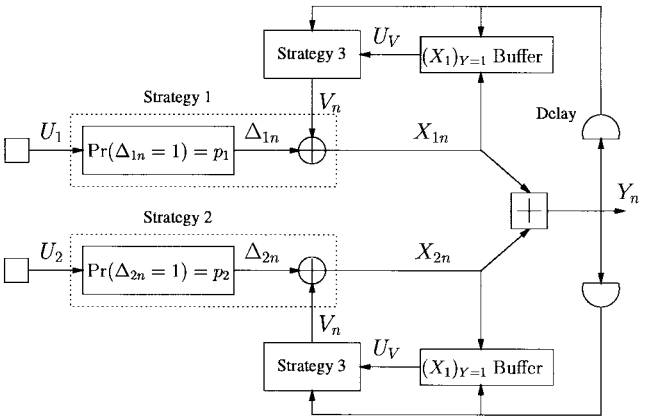


Fig. 1. A simple strategy for the noiseless binary adder channel with feedback. The ambiguous X_{1n} form the source sequence U_V.

This correspondence is organized as follows. In Section II we consider the BAC-FB in detail to motivate the final structure of our strategies. We follow an approach similar to Vinck's [5], [6] to show that the equal-rate point on the boundary of the capacity region of the BAC-FB can be approached with a simple feedback strategy. For the nonequal rate points, we show that it is important to add a data compression algorithm to the strategy. In Section III we modify the structure of our strategies to approach any rate point in the capacity region of channels for which $H(X_1|X_2Y)=0$ or $H(X_2|X_1Y)=0$. This structure is applied to the binary switching multiple-access channel. Finally, Section IV concludes the correspondence with some remarks.

Throughout the correspondence, the description of our strategies and the "proof" that they operate correctly will be asymptotic. The derivation of bounds on the error probability for finite message lengths is somewhat tedious, but straightforward as long as one is not concerned with finding the *best* possible bounds. An example of such a derivation can be found in Appendix A.

II. Strategies for the Binary Adder Channel

The first strategy we shall consider is depicted in Fig. 1. The two users independently generate the information bit sequences U_1 and U_2 and encode these into sequences Δ_1 and Δ_2 of independent and identically distributed bits so that $\Pr(\Delta_{1n}=1)=p_1$ and $\Pr(\Delta_{2n}=1)=p_2$, all n. For example, User 1 (and User 2) can map the information bits U_1 (and U_2) onto a point in the unit interval as in [12], and then divide this interval into one part with *a priori* probability p_1 (and p_2) and another with *a priori* probability $1-p_1$ (and $1-p_2$). The bits Δ_{1n} and Δ_{2n} are then added modulo 2 to V_n to obtain the inputs X_{1n} and X_{2n} to the channel. The output Y_n of the channel is fed back to both transmitters (with a small delay) so that both know X_{1n} and X_{2n} after receiving the feedback Y_n. The receiver, however, does not know X_{1n} or X_{2n} when $Y_n=1$. To resolve these ambiguities, whenever $Y_n=1$ the transmitters append X_{1n}, which both know, to a bit stream U_V and encode this stream

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into the sequence of bits V by using a single-user feedback strategy, e.g., [9]–[14].

A. The Equal-Rate Point

We first consider the equal-rate points of the BAC-FB capacity region. For this case we set $p_1 = p_2 = p$ and the analysis of the strategy becomes particularly simple. As long as the buffer for the ambiguous X_1 is stable, the transmitters can resolve all the ambiguities. But the rate at which bits enter this buffer is $\Pr(Y_n = 1)$ and we can empty the buffer at the capacity $C(p)$ of the channel from V to Y. Thus, the condition for buffer stability is simply $\Pr(Y_n = 1) < C(p)$. It is easily checked that $\Pr(Y_n = 1) = 2p(1-p)$ and that the channel from V to Y is a binary-symmetric erasure channel (BSEC) with capacity

$$C(p) = \left[p^2 + (1-p)^2\right] \cdot \left[1 - h\left(\frac{p^2}{p^2 + (1-p)^2}\right)\right] \quad (5)$$

where $h(p) = -p\log_2(p) - (1-p)\log_2(1-p)$ is the binary entropy function (see also Vinck [5]). We find that $2p(1-p) < C(p)$ when $0 \le p < p^* = 0.23766$, which limits the transmission rates of the users to values less than $h(p^*) = 0.7911$. It follows that as long as the strategy used to transmit the V approaches the capacity of the V to Y channel, one can approach equal-rate pairs up to $(0.7911, 0.7911)$.

To show that the condition $2p(1-p) = C(p)$ exactly determines the equal-rate point on the boundary of the capacity region, we must consider a result of Willems [4], who showed that the equal-rate boundary point is $(h(p), h(p))$ where p satisfies $0 \le p \le 1/2$ and

$$h(p) = \tfrac{1}{2}\left[h(2p(1-p)) + p^2 + (1-p)^2\right]. \quad (6)$$

But if we expand the right side of (5) and set $\pi = p^2 + (1-p)^2$ we obtain

$$\begin{aligned} C(p) &= \pi + p^2\log_2(p^2/\pi) + (1-p)^2\log_2((1-p)^2/\pi) \\ &= \pi + p^2\log_2(p^2) + (1-p)^2\log_2((1-p)^2) - \pi\log_2(\pi) \\ &\quad - (1-\pi)\log_2(1-\pi) + (1-\pi)\log_2(1-\pi) \\ &= \pi + 2p^2\log_2(p) + 2(1-p)^2\log_2(1-p) \\ &\quad + 2p(1-p)\log_2(2p(1-p)) + h(\pi) \\ &= p^2 + (1-p)^2 + 2p(1-p) - 2h(p) + h(2p(1-p)) \end{aligned} \quad (7)$$

where the third and fourth steps follow from $1 - \pi = 2p(1-p)$. Setting $C(p) = 2p(1-p)$ we obtain (6), as desired.

B. Other Rate Points

We next consider the case $p_1 \neq p_2$ for which $\Pr(Y_n = 1) = p_1(1-p_2) + p_2(1-p_1)$ and the V-to-Y channel is a binary-symmetric erasure channel with capacity $C(p_1, p_2)$ given by

$$[p_1p_2 + (1-p_1)(1-p_2)] \cdot \left[1 - h\left(\frac{p_1p_2}{p_1p_2 + (1-p_1)(1-p_2)}\right)\right]. \quad (8)$$

The approachable rate points are those $(h(p_1), h(p_2))$ where (p_1, p_2) satisfies $\Pr(Y_n = 1) \le C(p_1, p_2)$, and they are plotted in Fig. 2. We see that the strategy of Fig. 1 approaches the equal-rate point on the boundary of the capacity region, but it does *not* do so for any other (nontrivial) point on the boundary. For example, consider the boundary point $(1, 0.5)$. In this case, we need $h(p_1) = 1$ and $h(p_2) = 0.5$, or $p_1 = 0.5$ and $p_2 \approx 0.11$. For these values, $\Pr(Y_n = 1) = 0.5$ but $C(0.5, 0.11) = 0.25$.

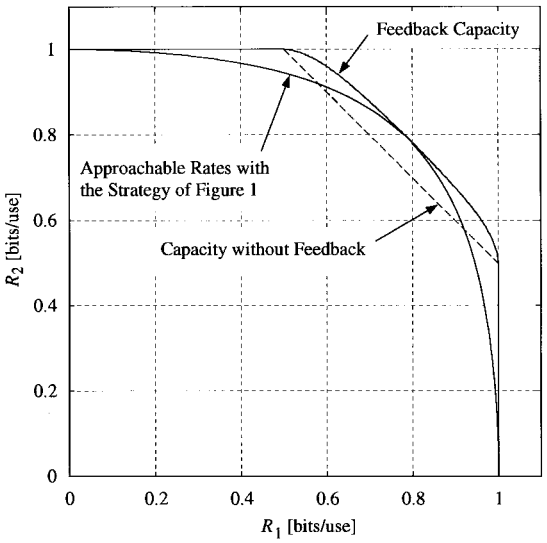


Fig. 2. Rates approachable with error probability approaching zero for the BAC-FB using the strategy of Fig. 1.

C. Adding Source Coding

The reason that the strategy of Fig. 1 does not approach all capacity points becomes apparent when we examine the uncertainty that the receiver has about X_{1n} when $Y_n = 1$. This uncertainty, after decoding V, is

$$H(X_1|V, Y = 1) = h\left(\frac{p_1(1-p_2)}{p_1(1-p_2) + (1-p_1)p_2}\right). \quad (9)$$

For example, if $p_1 = p_2$ we obtain $H(X_1|V, Y = 1) = 1$, so that U_V is a sequence of uniformly random bits as far as the receiver is concerned. But if $p_1 = 0.5$ and $p_2 \approx 0.11$ then $H(X_1|V, Y = 1) = 0.5$. Thus it is not surprising that the strategy of Fig. 1 does not approach the rate point $(1, 0.5)$, as one is trying to send a binary sequence U_V which has entropy 0.5 bit per symbol without compressing it first.

We thus modify the encoding structure by adding a source encoder in front of both users' Strategy 3 boxes. This source encoder must be capable of compressing a binary source to entropy, e.g., one could use an arithmetic source encoder [15, p. 61]. The resulting set of approachable rates matches those obtained in (1)–(4) for a binary V and $\Pr(X_1 \neq V) = p_1$, $\Pr(X_2 \neq V) = p_2$. However, we do not know whether this $p_{VX_1X_2Y}$ distribution approaches all the rate points in the BAC-FB capacity region. Thus we prefer to introduce a more general strategy that will also allow us to generalize our approach to other channels.

III. Strategies for $H(X_1|X_2Y) = 0$ Channels

The next structure we shall consider is depicted in Fig. 3. We explain how the various boxes in Fig. 3 operate. Users 1 and 2's "X_1 Buffer" box collects the sequence of X_{1n}. The contents of this buffer are passed to the "Compress" box that contains a source-coding scheme which can compress a discrete memoryless source to its entropy given the side information $V_n = v_n$ and $Y_n = y_n$, e.g., an arithmetic source encoder. This approach is similar to that described in [9]–[11] in the sense that some symbols are retransmitted after a compression. The source encoder output is the sequence of bits U_V.

The "Strategy 3" box contains a single-user feedback strategy that can approach the rate $I(V; Y)$ on the V-to-Y channel when the probability distribution $p_V(\cdot)$ is fixed. One may, e.g., use the Horstein-like strategies of [12]–[14], but as far as we know no one has

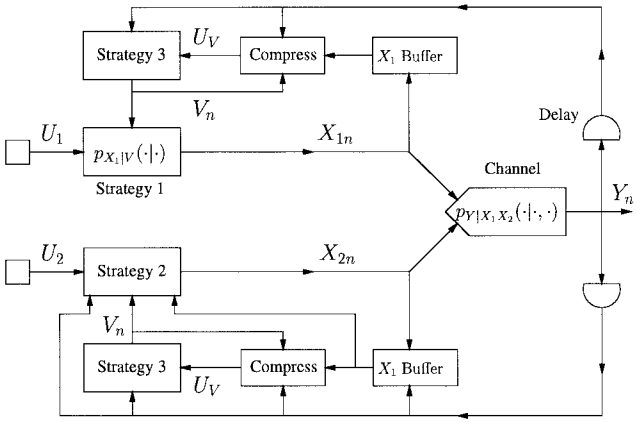


Fig. 3. An encoding structure for channels for which $H(X_1|X_2Y) = 0$. The input X_{1n} is formed from U_1 and V_n. User 2 forms X_{2n} by using a single-user feedback strategy for which V_n and X_{1n} are assumed known to the receiver.

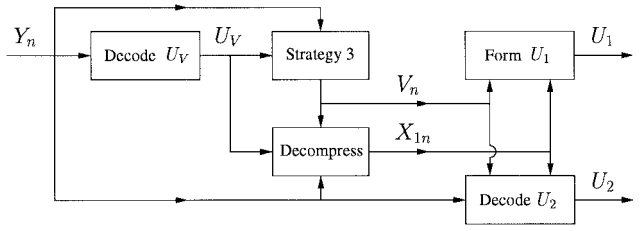


Fig. 4. The receiver for the encoding structure of Fig. 3.

proved that these approach $I(V;Y)$ for arbitrary discrete memoryless channels (DMC's)—even though it seems clear that they do. Of course, one may always resort to the source coding strategies of [9]–[11].

The "$p_{X_1|V}(\cdot|\cdot)$" box (Strategy 1) operates like Horstein's strategy but without feedback. For example, if X_1 takes on K_1 values, then the "$p_{X_1|V}(\cdot|\cdot)$" box maps the information bits U_1 onto a point in the unit interval and divides this interval into K_1 subintervals with *a priori* probabilities specified by $p_{X_1|V}(\cdot|v_n)$. The channel input X_{1n} is then chosen according to the subinterval in which U_1 lies.

The box labeled "Strategy 2" operates like "Strategy 3" except that it assumes that the receiver has decoded the U_V sequence and formed the V and X_1 sequences. Once the receiver knows V and X_1, the information rate of the resulting U_2-to-Y channel is $I(X_2;Y|X_1V)$, as required by (2).

The receiver (Fig. 4) first decodes the U_V sequence, from which the V sequence can be generated using the Y. Using the V and Y sequences, the receiver can decompress U_V to form the X_1. The bit sequence U_1 is obtained from the V and X_1 sequences. Finally, the U_2 sequence is decoded by using the V, X_1, and Y.

A. A Condition for Achievable Rates

Using the strategy of Fig. 3, we find that the U_V buffer will tend to empty as long as the senders can, at rate $I(V;Y)$, resolve the receiver's ambiguities, which have rate $H(X_1|YV)$. The condition for the U_V buffer to be stable is thus

$$H(X_1|YV) < I(V;Y). \tag{10}$$

Rewriting this condition by adding $I(X_1X_2;Y|V)$ to both sides of (10) and simplifying using $H(X_1|X_2V) = H(X_1|V)$ and $H(X_1|X_2Y) = 0$, we obtain the condition

$$I(X_1;Y|X_2V) + I(X_2;Y|X_1V) < I(X_1X_2;Y). \tag{11}$$

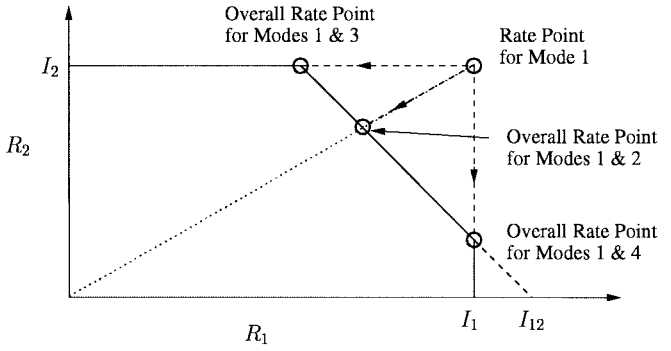


Fig. 5. The rate region defined by the bounds (1)–(3) for a particular $p_{VX_1X_2Y}$. In this figure, $I_1 = I(X_1;Y|X_2V)$, $I_2 = I(X_2;Y|X_1V)$, and $I_{12} = I(X_1X_2;Y)$. The operating points for the modes of operation of Fig. 6 are circled.

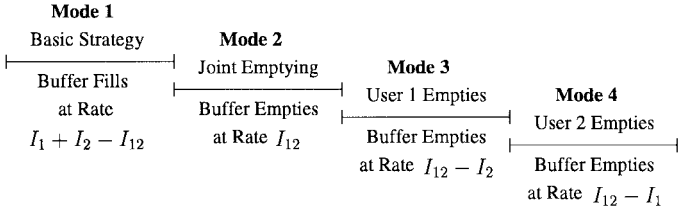


Fig. 6. Four modes of operation to approach any point in the rate region shown in Fig. 5.

Thus, for all $p_{VX_1X_2Y}$ for which the sum-rate bound (3) is unnecessary in the bounds (1)–(3), the strategy of Fig. 3 approaches all rate points in the rectangle defined by (1) and (2). For example, all the rate points achieved by the strategy of Section II-C obey (11). Those $p_{VX_1X_2Y}$ which do not satisfy (11) are treated next.

B. Mixed Strategies

For those $p_{VX_1X_2Y}$ for which the condition (11) is not satisfied, the bounds (1)–(3) determine a pentagon as in Fig. 5. As in this figure, we will use the shorthand I_1, I_2, and I_{12} for $I(X_1;Y|X_2V)$, $I(X_2;Y|X_1V)$, and $I(X_1X_2;Y)$, respectively.

We consider the queueing rate R_Q for the U_V buffer, i.e., the arrival rate minus the departure rate. From (10) and (11), we have

$$R_Q = I_1 + I_2 - I_{12}. \tag{12}$$

Suppose that in a first mode of operation both senders send their data in a block of length N ("Mode 1" in Fig. 6). Then after this block of data has been sent, the U_V buffer will have about R_QN bits in it. The users can now, instead of sending new data in the next block, empty the buffer by sending at the "superuser" rate I_{12}, i.e., they switch to a transmission mode dedicated to emptying the U_V buffer ("Mode 2" in Fig. 6). In fact, if the capacity of the superuser channel is larger than $I_{12} = I(X_1X_2;Y)$, the users will be able to transmit at a higher rate than that specified by the bounds (1)–(3). For example, this occurs in the Gaarder–Wolf scheme [16] when the users are in the mode of cooperatively reconciling ambiguities at the receiver. The overall information rates for this dual-mode strategy are

$$R_1 = \frac{\#\text{Bits}}{\#\text{Uses}} = \frac{N \cdot I_1}{N + NR_Q/I_{12}} = \frac{I_{12}}{I_1 + I_2} \cdot I_1 \tag{13}$$

$$R_2 = \frac{I_{12}}{I_1 + I_2} \cdot I_2. \tag{14}$$

This rate point is labeled in Fig. 5 as the "overall rate point for Modes 1 and 2." We remark that Vinck [6] used a dual-mode strategy to empty the U_V buffer for approaching the total cooperation line of the noiseless ternary adder channel with a *binary* V. But one can also use

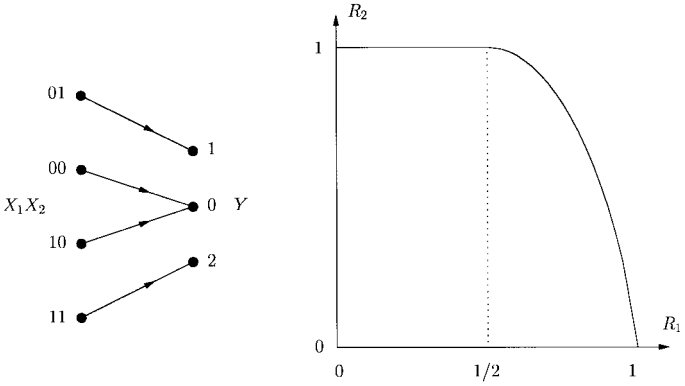


Fig. 7. The two-user binary switching multiple-access channel and its capacity region with and without feedback.

a *ternary* V to approach the total cooperation line without resorting to two modes of operation, as was shown by Lalive d'Epinay and Giger [17]. In fact, any rate point approached by time-sharing the modes of operation in Fig. 6 can be approached by using only Mode 1 with a new V which has the time sharing built into it. This is shown in Appendix B.

The above demonstrates that one can approach one of the points on the boundary of the pentagon with a dual-mode strategy. We now show that the outermost corner points of the pentagon can also be so approached by a dual-mode strategy. The idea is simply to let *one* of the users empty the U_V buffer in the second block ("Mode 3" or "Mode 4" in Fig. 6) while the other user continues to send data as in the first block and ignores the buffer bits that the other user is sending (Mode 1). For example, if during the second block User 1 empties the U_V buffer by himself (Mode 3) while User 2 is sending data (Mode 1), the U_V buffer empties at rate I_1 but is simultaneously filling up at rate R_Q. Thus User 1's overall information rate is

$$R_1 = \frac{\#\text{Bits}}{\#\text{Uses}} = \frac{N \cdot I_1}{N + NR_Q/(I_1 - R_Q)} = I_{12} - I_2 \qquad (15)$$

while $R_2 = I_2$. The resulting rate point is labeled in Fig. 5 as the "overall rate point for Modes 1 and 3." The "overall rate point for Modes 1 and 4" is obtained by interchanging the roles of Users 1 and 2.

The above shows that any point on the diagonal line of the pentagon in Fig. 5 can be approached: one simply time-shares the four modes of operation in Fig. 6. This, along with the result of Section III-A, shows that the structure of Fig. 3 used in several modes can approach any point in the rate region $\mathcal{R}^{\mathrm{CL}}$. Thus this strategy can approach any point in the capacity region of channels with $H(X_1|X_2Y) = 0$.

C. The Binary Switching Multiple-Access Channel

The binary switching multiple-access channel (BS-MAC) is an example of a channel with $H(X_2|X_1Y) = 0$ but $H(X_1|X_2Y) > 0$ (we have exchanged the roles of X_1 and X_2 here to conform with [16] and [18]). Both users' input alphabets are $\{0, 1\}$ and the output is $Y = (X_1 + X_2) \cdot X_2$ (see Fig. 7). The capacity region $\mathcal{C}_{\mathrm{FB}}$ of this channel with and without feedback is the same, and is the set of rate pairs (R_1, R_2) for which [6]

$$0 \le R_2 \le \begin{cases} 1, & \text{for } 0 \le R_1 \le 1/2 \\ h(R_1), & \text{for } 1/2 < R_1 \le 1. \end{cases} \qquad (16)$$

A plot of the capacity region is given in Fig. 7. In fact, the zero-error capacity region of the BS-MAC (with and without feedback) also coincides with $\mathcal{C}_{\mathrm{FB}}$ [18].

The strategy of Fig. 3, with the roles of Users 1 and 2 reversed, is particularly simple for this channel. User 2 codes as in Fig. 1 by converting the information bit stream U_2 into a sequence $\Delta_2 = X_2$ of independent and identically distributed bits where $\Pr(X_{2n} = 0) = p$, all n. User 1 then sees a binary erasure channel (BEC) with erasure probability p, and makes Strategy 2 a selective-repeat automatic repeat request (ARQ) strategy to approach the capacity $1 - p$ of the BEC. Strategy 3 is not used at all because $H(X_2|Y) = 0$. The approachable rate region is thus the convex hull of the points $(0,0), (0,1)$, and those (R_1, R_2) for which $0 \le p \le 1/2$. This coincides with the capacity region $\mathcal{C}_{\mathrm{FB}}$. This example demonstrates the importance of making Strategy 2 in Fig. 3 a *feedback strategy*, and not simply a "$p_{X_1|V}(\cdot|\cdot)$" box as for User 1.

IV. Concluding Remarks

Other simple strategies are included in the structure of Fig. 3 that were not explicitly mentioned in Sections II and III. For example, for the BAC-FB, a simpler strategy to approach $(R_1, R_2) = (1, 0.5)$ than that of Section II-C is to let User 1 send data while User 2 uses an ARQ strategy. This strategy is included in the structure of Fig. 3 by letting Strategy 2 do the ambiguity resolving and not using Strategy 3 at all. In fact, this strategy is basically the same as using Modes 1 and 4 in Fig. 6. The structure of Fig. 3 includes similar strategies for other channels as well.

The MAC's for which $H(X_1|X_2Y) = 0$ or $H(X_2|X_1Y) = 0$ are not the only MAC-FB's for which the capacity region is known to be the rate region of Cover and Leung. For example, it is known that $\mathcal{R}^{\mathrm{CL}}$ is the capacity region of the channels considered in this correspondence but with feedback to User 2 only [19]–[21]. It does not seem that the strategies presented here can be extended to achieve all points inside the capacity region of these other channels. The difficulty is that the structure of Fig. 3 relies heavily on User 2 being able to form the U_V sequence in a simple manner, i.e., without decoding a code. Another class of channels for which $\mathcal{R}^{\mathrm{CL}}$ is the capacity region was found by Hekstra and Willems [22, Sec. VII].

An interesting problem is whether the strategies presented here can be extended to achieve all points inside $\mathcal{R}^{\mathrm{CL}}$ for arbitrary (full feedback) MAC-FB's. A preliminary study of this question suggests that this is indeed possible. A further problem is the design of feedback strategies achieving rate points outside $\mathcal{R}^{\mathrm{CL}}$, e.g., by using the *directed information* characterization of the MAC-FB capacity region [23].

Finally, we note that there is a simple strategy which achieves all points inside the capacity region of the additive white Gaussian noise MAC-FB [24]. It would be most useful if a similar strategy achieved all points inside the capacity region of the general MAC-FB.

Appendix A
Bounding the Error Probability

The analysis of the strategies has up to this point been asymptotic. We now show how one can bound the error probability for *finite* message lengths by considering the structure of Fig. 1 and the equal-rate point of the BAC-FB. The analysis for other rate points and for other channels will be complicated by the source encoder of Strategy 3. However, for these cases one can still proceed along the same lines presented here.

We will transmit data in blocks like the block Markov superposition technique of [2]. In each block, B_1 information bits are sent in $N+1$ channel uses. Consider Strategies 1 and 2 with $p = p_1 = p_2$ and $0 < p < 0.5$. One would expect to have to set $B_1 \approx h(p) \cdot N$ for both strategies. If $\Delta_{1n} = 0$ occurs k times in the first N channel uses, the final interval of Strategy 1 will have size $(1-p)^k p^{N-k}$.

As long as this interval has a size smaller than 2^{-B_1}, we will need only one more channel use to specify the correct message interval (the final length $(1-p)^k p^{N-k}$ interval will usually overlap two message intervals). We thus say that the error event $\mathcal{E}_1$ occurs if $(1-p)^k p^{N-k} \geq 2^{-B_1}$. Equivalently, $\mathcal{E}_1$ occurs if

$$\frac{k}{N} \geq \frac{-R_1 - \log(p)}{\log((1-p)/p)} \tag{17}$$

where $R_1 = B_1/N$. To simplify the expressions, we set

$$\lambda = (-R_1 - \log(p))/(\log(1-p)/p)$$

and

$$E_1(R_1, p) = -h(\lambda) - \lambda \log(1-p) - (1-\lambda)\log(p).$$

We then use [25, Lemma 4.7.2] to write

$$\Pr(\mathcal{E}_1) \leq 2^{-NE_1(R_1, p)} \tag{18}$$

for $1-p < \lambda \leq 1$ ($\lambda = 1$ is also admissible in [25, eq. (4.7.2)]). One can check that $1-p < \lambda$ as long as $R_1 < h(p)$, and that $E_1(R_1, p)$ is positive for these values. The bound $\lambda \leq 1$ is automatically satisfied because the size of the final interval is at most $(1-p)^N$. Interpreting the bound (18), as long as $R_1 < h(p)$ the probability of $\mathcal{E}_1$ can be made to vanish by letting $N \to \infty$. The error exponent for Strategy 2 is also $E_1(R_1, p)$.

Next, consider the event $\mathcal{E}_B$ that the $(X_1)_{Y=1}$ buffer will overflow for the bth block. Let L_B be the buffer size. Since $\Pr(Y_n = 1) \approx 2p(1-p)$ we expect to have to set $L_B \approx 2p(1-p) \cdot N$. Letting k be the number of $Y_n = 1$ in the $N+1$ channel uses of the bth block, a buffer overflow occurs if $k \geq L_B$. Setting $\lambda = L_B/(N+1)$ and

$$E_B(L_B, p) = -h(\lambda) - \lambda \log(2p(1-p)) - (1-\lambda)\log(1-2p(1-p))$$

we can again use [25, Lemma 4.7.2] to bound

$$\Pr(\mathcal{E}_B) \leq 2^{-(N+1)E_B(L_B, p)}. \tag{19}$$

Finally, consider Strategy 3 for the BSEC from V to Y. Let $\mathcal{E}_3$ be the event that a block error is made for the $N+1$ channel uses in the bth block. There are (fixed block length) feedback strategies that can attain the random coding exponent for the BSC [14]. It is easy to show that the same strategies obtain the random coding exponent for the BSEC, i.e.,

$$\Pr(\mathcal{E}_3) \leq 2^{-(N+1)E_3(R_V, p)} \tag{20}$$

where (see [26, Ch. 5])

$$E_3(R_V, p) = \max_{\rho} \left\{ -\log_2 \left[2^{-\rho} \left[(1-p)^{2/(1+\rho)} + p^{2/(1+\rho)} \right]^{1+\rho} + 2p(1-p) \right] - \rho R_V \right\} \tag{21}$$

and where R_V is the rate of Strategy 3. The exponent $E_3(R_V, p)$ is positive as long as R_V is less than the BSEC capacity $C(p)$ given by (5). We further require that $L_B \leq R_V \cdot (N+1)$ so that Strategy 3 can empty the $(X_1)_{Y=1}$ buffer.

We say that an error $\mathcal{E}$ occurs if any of the error events $\mathcal{E}_1$, $\mathcal{E}_B$, or $\mathcal{E}_3$ occurs in any of the $B+1$ blocks of transmission. Following the approach of [2], the first transmission block requires only the bounds (18) and (19) and the last block requires only the bound (20). All other blocks require all three bounds. Combining the $4 \cdot B$ bounds using the union bound [27, p. 264] we have

$$\Pr(\mathcal{E}) \leq B \left\{ 2 \cdot 2^{-NE_1(R_1, p)} + 2^{-(N+1)E_B(L_B, p)} + 2^{-(N+1)E_3(R_V, p)} \right\} \tag{22}$$

where the extra factor of 2 is for bounding the error probabilities of Strategies 1 and 2. The overall transmission rate for both users is

$$R = \frac{B}{B+1} \cdot \frac{N}{N+1} \cdot R_1. \tag{23}$$

It can be shown that (22) and (23) guarantee that the users can send reliably at rates arbitrarily close to the point $(R_1, R_2) = (0.7911, 0.7911)$ on the boundary of the BAC-FB capacity region. For example, suppose that each user wants to achieve the rate $R = 0.78$ at $\Pr(\mathcal{E}) = 10^{-6}$ by transmitting in $B+1 = 200$ blocks. We ask what N is required for each block.

Consider first $E_1(R_1, p)$. From (23) we obtain

$$NR_1/(N+1) \geq 0.784.$$

We choose a target value of $R_1 = 0.785$ so that $N \geq 784$ (we will see shortly that N must be much larger to attain the desired error probability). We next choose $p = 0.2357$ so that $h(p) \approx 0.7878$ is larger than R_1. The resulting error exponent is $E_1(R_1, p) = 1.1 \cdot 10^{-5}$.

Next, we must choose L_B small enough so that R_V can remain small. We set $L_B = 1.01 \cdot 2p(1-p)(N+1)$ to obtain

$$E_B(L_B, p) = 4.1 \cdot 10^{-5}.$$

Finally, we choose $R_V = 1.01 \cdot 2p(1-p) = 0.3603$. Since $C(p) = 0.3673$ is larger than this value, we can obtain a positive $E_3(R_V, p)$. We find that

$$E_3(R_V, p) \geq 1.1 \cdot 10^{-5}$$

by using $\rho = 0.01$.

Combining the bounds, we see that $\Pr(\mathcal{E}_1)$ and $\Pr(\mathcal{E}_3)$ will dominate (22). We thus weaken the bound (22) to

$$\Pr(\mathcal{E}) \leq 199 \cdot 4 \cdot 2^{-N \cdot (1.1 \cdot 10^{-5})}. \tag{24}$$

This bound guarantees that $\Pr(\mathcal{E}) \leq 10^{-6}$ if $N \geq 2.7 \cdot 10^6$. Thus one must send about 540 million symbols in 200 blocks. The overall rate given by (23) is $R = 0.781$.

Appendix B
Approaching Capacity with a Single Mode of Operation

The four modes of Fig. 6 can be combined into a single mode by increasing the cardinality of the random variable V as follows. Given $p_{VX_1X_2Y}$, define the four random variables

$$\begin{aligned} V^{(1)} &= V \\ V^{(2)} &= [V \ \ X_1 \ \ X_2] \\ V^{(3)} &= [V \ \ X_1] \\ V^{(4)} &= [V \ \ X_2]. \end{aligned}$$

We replace V with the random variable $\tilde{V}$ where $\tilde{V} = V^{(m)}$ with probability t_m, $m = 1, 2, 3, 4$, and $\sum_{m=1}^{4} t_m = 1$. Note that

$$p_{X_1X_2|V^{(m)}} = p_{X_1|V^{(m)}} \cdot p_{X_2|V^{(m)}}$$

for every $V^{(m)}$, so that

$$p_{X_1X_2|\tilde{V}} = p_{X_1|\tilde{V}} \cdot p_{X_2|\tilde{V}}$$

as required by (4).

We next show that $V^{(m)}$ corresponds to Mode m in Fig. 6. Using $\tilde{V}$ rather than V does not change $p_{X_1X_2Y}$, so that the sum-rate bound $I(X_1X_2; Y)$ is not changed. If $\tilde{V} = V^{(2)}$ then $I(X_1; Y|X_2\tilde{V}) = 0$ and $I(X_2; Y|X_1\tilde{V}) = 0$. Similarly, if $\tilde{V} = V^{(3)}$ then $I(X_1; Y|X_2\tilde{V}) = 0$ and $I(X_2; Y|X_1\tilde{V}) = I(X_2; Y|X_1V)$, and if $\tilde{V} = V^{(4)}$ then $I(X_1; Y|X_2\tilde{V}) = I(X_1; Y|X_2V)$ and

$I(X_2; Y|X_1\tilde{V}) = 0$. Thus the bounds (1) and (2) with V replaced by $\tilde{V}$ define a rectangle whose maximum sum-rate point is at

$$(R_1, R_2) = ((t_1 + t_4)I_1, (t_1 + t_3)I_2) \tag{25}$$

where $I_1 = I(X_1; Y|X_2V)$ and $I_2 = I(X_2; Y|X_1V)$. One can make the rate pair in (25) any rate point inside the pentagon of Fig. 5 by choosing the "time-sharing" probabilities t_m, $m = 1, 2, 3, 4$ appropriately. For example, if $I_1 = 1$, $I_2 = 2$, $I_{12} = 2.5$, and we want to obtain $(R_1, R_2) = (0.75, 1.75)$ we can choose $t_1 = 0.75$, $t_2 = 0.125$, $t_3 = 0.125$, and $t_4 = 0$. Because the sum-rate bound (3) is superfluous for these rate points, one can achieve them with Mode 1 in Fig. 6.

We may interpret the result of this appendix as follows: for any point inside the capacity regions of the MAC-FB's considered here, there exists a distribution $p_{VX_1X_2Y}$ so that the sum rate bound (3) is unnecessary.

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Hadamard Transforms of Three-Term Sequences

Guang Gong and Solomon W. Golomb, *Fellow, IEEE*

***Abstract*— Certain three-term sequences of period $2^n - 1$ are conjectured to have the two-level autocorrelation property. In this note, a formula involving Hadamard transforms of three-term sequences is presented. Its validity has been verified by computer on the range of $5 \le n \le 23$.**

***Index Terms*—Hadamard transform, *m*-sequences, two-level autocorrelation sequences.**

I. CONJECTURE

Let $n = 2m + 1$, $q = 1 + 2^m$, and α be a primitive element of GF(2^n). Let $f(x) = \mathrm{Tr}\,(x + x^q + x^{q^2})$ and $A = \{a_i\}$ where $a_i = f(\alpha^i)$. In [1], the sequence A is conjectured to have a two-level

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D. Gong is with the Center for Applied Cryptographic Research, Department of Combinatorics and Optimization, University of Waterloo, Waterloo Ont., Canada N2L 3G1.

S. W. Golomb is with The Communication Sciences Institute, University of Southern California, Electrical Engineering-Systems, EEB #500, Los Angeles, CA 90089-2565 USA.

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