Reliable Communication in Networks with Multi-access Interference

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***Abstract*— Achievable rates are derived for reliable communication in networks with multi-access interference. Capacity is established for some such networks. A network flow formulation is developed that includes rate gains and losses caused by correlating the multi-access channel inputs.**

I. Introduction

We consider networks with multiple access interference. Such networks are represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex (or node) set $\mathcal{V}$ and directed-edge set $\mathcal{E}$. We associate one "input" random variable X_e with every edge e, and one "output" random variable Y_u with every node u that is a noisy version of the X_e whose edge e ends at u. In other words, every Y_u is the output of a multi-access channel (MAC) taking inputs X_e of the edges ending at u. For simplicity, we often consider only one MAC at a time, in which case we write its output and inputs as Y and $X_1, X_2, \ldots, X_K$, respectively, so the MAC is the conditional probability distribution $p(y|x_1, x_2, \ldots, x_K)$. In the case of a deterministic MAC, the output is $y = f(x_1, \ldots x_K)$ where $f(\cdot)$ is some function.

The capacity of networks with MACs is not known in general even for a single source-destination pair. A special model considered in [1] (see also [2]–[4]) is the finite field multiple access network. There, every MAC puts out

$$Y = \alpha_1 X_1 + \ldots + \alpha_K X_K + Z \tag{1}$$

where the X_i, $i = 1, 2, \ldots, K$, take on values in the finite field $\mathbb{F}_q$ with q elements, $\alpha_i \in \mathbb{F}_q \backslash 0$, and the noise Z, $Z \in \mathbb{F}_q$, is independent of the X_i. The multicast capacity of such networks is given by a cut-set bound [5]. It turns out that the optimal per-letter joint distribution of the $X_1, X_2, \ldots, X_K$ is a uniform product distribution; this is in part due to the symmetrizing properties of the finite field. Thus, for this particular model we do not gain by "correlating" the inputs. More generally, it is not known if there exists a simple characterization of the multicast capacity, e.g., as in [6] where the network edges are non-interfering point-to-point links, and [7] where the network consists of deterministic broadcast channels with no interference.

We extend existing results in three directions. First, we establish an achievable rate for a single unicast or multicast connection. It is, for some networks, strictly smaller than the standard cut-set rate. Second, we give a converse for classes of "diamond" networks with one MAC. Third, we use a flow formulation to extend our achievable rates to networks of MACs. While it is unlikely that the flow extension achieves capacity in general, the results are a step towards understanding the capacity of wireless networks where MAC interference limits rates.

Our coding method is related to the study of MACs with correlated sources. The paper [8] gives a single-letter achievable rate region for such channels, and also a multi-letter converse, i.e., the converse is in terms of mutual information between codewords as opposed to alphabet symbols. As [9] demonstrates by means of a counterexample, the achievable region and the converse are not the same. An interesting single-letter bound on the sum-rate of the MAC with correlated sources is given in [10], but unfortunately this bound is not the capacity in general.

This paper is organized as follows. In Section II, we motivate the problem by studying a simple, and yet instructive, example of a two-relay "diamond" network. In Section III, we briefly review facts about typical sequences that we need for the proof of our main result in Section IV. In Section V, we discuss general networks with multi-access interference. Section VI concludes the paper.

II. Introductory Example: A Two Relay Network

Consider the network in Fig. 1 that has a source S communicating to a destination Y via two relays. This setup is studied in [11] where there is a broadcast channel between the source and the relays X_1 and X_2, and a MAC with inputs X_1 and X_2 and output Y. The capacity of this setup is not known in general. To simplify the problem, we replace the broadcast channel from the sender to the relays with two point-to-point channels with capacities C_1 and C_2. However, the problem still seems difficult because we might wish to use correlated MAC inputs X_1 and X_2.

A. An achievable rate

The following Theorem is implied by Theorem 3 that we prove in the next section.

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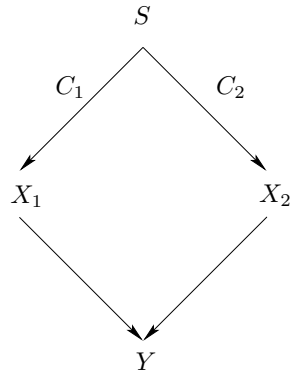


Fig. 1. A diamond network with a MAC. The channels from S to X_1 and X_2 are point-to-point channels with capacities C_1 and C_2.

Theorem 1: For the two relay network depicted in Fig. 1, the following rate is achievable

$$\begin{aligned} R = \max_{p(x_1,x_2)} \min(&C_1 + C_2 - I(X_1;X_2), \\ &I(X_1;Y|X_2) + C_2, \\ &C_1 + I(X_2;Y|X_1), \\ &I(X_1,X_2;Y)). \end{aligned} \tag{2}$$

Remark 1: For large C_1 and C_2, the minimum in (2) is

$$\max_{p(x_1,x_2)} I(X_1,X_2;Y) \tag{3}$$

which is the *collaborative capacity* of the MAC [5]. This is reasonable since, for sufficiently large C_1 and C_2, we can send the message to both relays and communicate at the collaborative capacity.

Remark 2: If we restrict attention to product distributions $p(x_1,x_2) = p(x_1)p(x_2)$, then one achieves the rate (2) for fixed $p(x_1)$ and $p(x_2)$ by splitting the source message into two independent messages, and sending the first message to relay X_1 and the second message to relay X_2. Our rate (2) thus represents a nontrivial regime of operation that lies between the above two extremes.

Remark 3: One can choose X_1, X_2, and Y to be length-L blocks of input symbols that we write as X_1^L, X_2^L, and Y^L, respectively. The following rate is then achievable

$$\begin{aligned} R = \max_{p(x_1^L,x_2^L)} \min(&C_1 + C_2 - \frac{1}{L} I(X_1^L;X_2^L), \\ &\frac{1}{L} I(X_1^L;Y^L|X_2^L) + C_2, \\ &C_1 + \frac{1}{L} I(X_2^L;Y^L|X_1^L), \\ &\frac{1}{L} I(X_1^L,X_2^L;Y^L)). \end{aligned} \tag{4}$$

It turns out that the limit of (4) as L becomes large gives the network capacity if the point-to-point links for the source are deterministic, as shown below.

B. Capacity for Deterministic Point-to-Point Channels

The cut-set bound [5, Ch. 14] for our diamond network is

$$\begin{aligned} R \le \max_{p(x_1,x_2)} \min(&C_1 + C_2, I(X_1;Y|X_2) + C_2, \\ &C_1 + I(X_2;Y|X_1), I(X_1,X_2;Y)). \end{aligned} \tag{5}$$

Note that the difference to (2) is only in the first term in (5).

Suppose our diamond networks has deterministic point-to-point links. We establish a tighter bound on the first term in (5), but unfortunately at the price of using a multi letter characterization. This is similar to the approach taken in [8]. We have the following result.

Theorem 2: One can reliably (i.e., with average block error probability approaching zero) communicate a message at rate R over the diamond network with deterministic point-to-point links if and only if R satisfies $0 \le R$ and

$$R \le C_1 + C_2 - \frac{1}{n} I(X_1^n;X_2^n) \tag{6}$$

$$R \le \frac{1}{n} I(X_1^n;Y^n|X_2^n) + C_2 \tag{7}$$

$$R \le C_1 + \frac{1}{n} I(X_2^n;Y^n|X_1^n) \tag{8}$$

$$R \le \frac{1}{n} I(X_1^n,X_2^n;Y^n) \tag{9}$$

for some distribution $p(x_1^n,x_2^n)$ where n is sufficiently large.

Proof: Achievability follows by Theorem 1. Conversely, for any reliably decodable code of length n, by Fano's inequality we have

$$nR \le I(X_1^n,X_2^n;Y^n) \tag{10}$$

$$\le H(X_1^n,X_2^n) \tag{11}$$

$$= H(X_1^n) + H(X_2^n) - I(X_1^n;X_2^n) \tag{12}$$

$$\le nC_1 + nC_2 - I(X_1^n;X_2^n) \tag{13}$$

where the last inequality is valid because X_1^n and X_2^n are functions of the point-to-point channel outputs. This gives (6); the bounds (7)-(9) follow by standard cut-set arguments. ■

C. Example 1.

Consider the network of Fig. 1 with the MAC

$$Y = X_1 + X_2 \tag{14}$$

where $\mathcal{X}_1 = \mathcal{X}_2 = \{0,1\}, \mathcal{Y} = \{0,1,2\}$, and the sum is over the integers. Let the capacities of the point-to-point links be $C_1 = C_2 = \frac{1}{2}\log_2 3 \approx 0.7925$. An upper bound on R is

$$I(X_1,X_2;Y) = H(Y) \le \log_2 3 \approx 1.585. \tag{15}$$

Furthermore, achieving $R = \log_2 3$ requires a uniform distribution on Y, which requires X_1 and X_2 to be correlated. Correlated inputs, on the other hand, require

$$C_1 + C_2 - I(X_1,X_2) = \log_2 3 - I(X_1,X_2) < \log_2 3. \tag{16}$$

It thus turns out that our code construction technique does not achieve the cut-set bound. A numerical solution of (2) yields

a maximal rate of $R = 1.5640$ where (X_1, X_2) distributed as follows:

$$p_{X_1X_2}(0,0) = p_{X_1X_2}(1,1) = 0.2925$$
$$p_{X_1X_2}(0,1) = p_{X_1X_2}(1,0) = 0.2075.$$

Furthermore, for this channel we do not gain rate by considering "2-letter" codes in addition to "single letter" codes.

We remark again that, for sufficiently large C_1 and C_2, e.g. for $C_1, C_2 \geq \log_2 3$, our problem reduces to finding the collaborative capacity. However, for the above constraints on the communication rates to both multiple access encoders, our problem cannot be reduced to this special case. We are facing a tradeoff: If we use highly correlated MAC inputs then we can maximize the information flow across the cut $(\{X_1, X_2\}, Y)$. But this reduces the information transfer across the cut $(S, \{X_1, X_2\})$ because $I(X_1; X_2)$ increases.

D. Example 2.

Consider again the network of Fig. 1 but where $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1, 2\}$ and

$$Y = \begin{cases} X_1 & \text{if } X_1 = X_2 \\ 3 & \text{if } X_1 \neq X_2. \end{cases} \tag{17}$$

Let the capacities of the point-to-point links be $C_1 = C_2 = 1$, so that the cut-set bound is $R \leq 2$. A numerical solution of (2) yields a maximal rate of $R = 1.861$ for an input distribution $p(x_1, x_2)$ of the form:

$p(x_1, x_2)$	$x_1 = 0$	1	2
$x_2 = 0$	0.1825	0.0754	0.0754
1	0.0754	0.1825	0.0754
2	0.0754	0.0754	0.1825

Due to the particular structure of this MAC, a high correlation between the inputs is desirable, and therefore the gap between our achievable rate and the cut-set bound is larger. As in the previous example, we do not gain by using "2-letter" codes.

III. Typical Sequences and Sets

We require some definitions and results on typical sets and sequences. We quote them from [5, Ch. 3, 4, and 14]. The typical set $T_\delta^n(X)$ with respect to the distribution $p(x)$ is the set of sequences $\underline{x} = (x_1, ..., x_n) \in \mathcal{X}^n$ with the property

$$2^{-n(H(X)+\delta)} < p(x_1, ..., x_n) < 2^{-n(H(X)-\delta)} \tag{18}$$

or

$$\left| -\frac{1}{n} \log p(\underline{x}) - H(X) \right| < \delta. \tag{19}$$

The size of the typical set satisfies

$$(1-\delta)2^{n(H(X)-\delta)} \leq |T_\delta^n(X)| \leq 2^{n(H(X)+\delta)}. \tag{20}$$

Similarly, the typical set $T_\delta^n(X, Y)$ of jointly typical sequences $(\underline{x}, \underline{y})$ with respect to the distribution $p(x, y)$ is defined as

$$\begin{aligned} T_\delta^n(X,Y) = \{&(\underline{x}, \underline{y}) \in \mathcal{X}^n \times \mathcal{Y}^n : \\ &\left| -\frac{1}{n} \log p(\underline{x}) - H(X) \right| < \delta, && (21) \\ &\left| -\frac{1}{n} \log p(\underline{y}) - H(Y) \right| < \delta, && (22) \\ &\left| -\frac{1}{n} \log p(\underline{x}, \underline{y}) - H(X,Y) \right| < \delta\}. && (23) \end{aligned}$$

The size of the typical set satisfies

$$(1-\delta)2^{n(H(X,Y)-\delta)} \leq |T_\delta^n(X,Y)| \leq 2^{n(H(X,Y)+\delta)}. \tag{24}$$

If we draw a sequence $(\underline{X}, \underline{Y})$ of length n according to $p(\underline{x}, \underline{y}) = \prod_{i=1}^n p(x_i, y_i)$, then

$$\mathrm{P}((\underline{X}, \underline{Y}) \in T_\delta^n(X,Y)) \to 1 \quad \text{as} \quad n \to \infty. \tag{25}$$

If we instead draw $(\widetilde{\underline{X}}, \widetilde{\underline{Y}})$ according to $\prod_{i=1}^n p(x_i)p(y_i)$, the probability that the resulting marginally typical sequences are also jointly typical is close to $2^{-nI(X;Y)}$, or more precisely

$$\begin{aligned} (1-\delta)2^{-n(I(X;Y)+3\delta)} &\leq \mathrm{P}((\widetilde{\underline{X}}, \widetilde{\underline{Y}}) \in T_\delta^n(X,Y)) \\ &\leq 2^{-n(I(X;Y)-3\delta)} \end{aligned} \tag{26}$$

for sufficiently large n. One can generalize the above definitions and results to any number of random variables in obvious ways.

IV. A Multiple Access Result

Consider a memoryless MAC $p(y|x_1, x_2)$. We wish to transmit a message M, $M \in \{1, ..., 2^{nR}\}$, that is seen by both encoders. We assign artificial sources messages M_1, $M_1 \in \{1, ..., 2^{nR_1}\}$, and M_2, $M_2 \in \{1, ..., 2^{nR_2}\}$, respectively, and define a mapping

$$f: \quad M \to (M_1, M_2) \tag{27}$$

that maps every message $m \in M$ to a pair of source messages such that M can be recovered from (M_1, M_2). At the source we use encoding functions

$$f_1: \quad M_1 \quad \to \quad X_1^n \tag{28}$$
$$f_2: \quad M_2 \quad \to \quad X_2^n \tag{29}$$

and at the destination a decoding function

$$g: \quad Y^n \to M. \tag{30}$$

Suppose we restrict the source messages M_1 and M_2 to take on 2^{nR_1} and 2^{nR_2} values, respectively. What is then the largest R for which we can transmit M reliably? We prove the following.

Theorem 3: If M_1 and M_2 take on 2^{nR_1} and 2^{nR_2} values, respectively, message M can be transmitted reliably if

$$\begin{aligned} R < \min(&R_1 + R_2 - I(X_1, X_2), I(X_1; Y|X_2) + R_2, \\ &R_1 + I(X_2; Y|X_1), I(X_1, X_2; Y)) \end{aligned} \tag{31}$$

for some distribution $p(x_1, x_2)$.

Proof: Our proof follows along the lines of the proof given in [5, Sec. 14.3.1]. The main difference is that we consider a joint distribution $p(x_1, x_2)$ on the inputs rather than a product distribution $p(x_1)p(x_2)$. A closely related idea appears in the paper [12, Thm. 2] that considers mismatched decoding for the MAC.

Codebook generation. Fix $p(x_1, x_2)$ and generate 2^{nR_1} independent codewords $\underline{X}_1(i)$, $i \in \{1, ..., 2^{nR_1}\}$, of block length n, where all components of each codeword are drawn independently and identically distributed (i.i.d.) with distribution $p(x_1)$. Similarly, generate 2^{nR_2} independent codewords $\underline{X}_2(j)$, $j \in \{1, ..., 2^{nR_2}\}$, with components drawn i.i.d. with distribution $p(x_2)$.

Encoding. Let $\widetilde{\mathcal{M}}$ be the set of all pairs (i, j) whose codewords are jointly typical, i.e., we have

$$\widetilde{\mathcal{M}} = \{(i, j) : (\underline{X}_1(i), \underline{X}_2(j)) \in T_\delta^n(X_1, X_2)\}. \tag{32}$$

The probability that any pair $(\underline{X}_1(i), \underline{X}_2(j))$ is jointly typical is bounded in (26), and we therefore have

$$|\widetilde{\mathcal{M}}| \geq (1 - \delta)\, 2^{n(R_1 + R_2 - I(X_1;X_2) - 3\delta)} \tag{33}$$

with high probability for large n. We can thus choose

$$R = R_1 + R_2 - I(X_1; X_2) - 3\delta + \frac{\log_2(1 - \delta)}{n} \tag{34}$$

for sufficiently large n while still ensuring that the mapping f is one-to-one.

Decoding. The receiver chooses the pair (i, j) such that

$$(\underline{x}_1(i), \underline{x}_2(j), \underline{y}) \in T_\delta^n(X_1, X_2, Y) \tag{35}$$

if such a pair exists and is unique, otherwise it declares an error. The index pair determines the overall message, since we have used a one-to-one function f to assign messages to index pairs.

Analysis of the error probability. Due to the symmetric code construction, the error probability is the same for all pairs of indices. We therefore assume that $(i, j) = (1, 1)$ was sent. We define the events

$$E_{ij} = \{(\underline{X}_1(i), \underline{X}_2(j), \underline{Y}) \in T_\delta^n(X_1, X_2, Y)\} \tag{36}$$

and bound the error probability as follows:

$$\begin{aligned} P_e &= \mathrm{P}(E_{11}^C \bigcup \cup_{(i,j)\neq(1,1)} E_{ij}) \\ &\leq \mathrm{P}(E_{11}^C) + \sum_{i\neq 1} \mathrm{P}(E_{i1}) \\ &\quad + \sum_{j\neq 1} \mathrm{P}(E_{1j}) + \sum_{i\neq 1, j\neq 1} \mathrm{P}(E_{ij}) \end{aligned} \tag{37}$$

where we have used the union bound. Following [5, (14.64)-(14.68)], for $i \neq 1$ we bound

$$\mathrm{P}(E_{i1}) = \mathrm{P}\left((\underline{X}_1(i), \underline{X}_2(1), \underline{Y}) \in T_\delta^n(X_1, X_2, Y)\right) \tag{38}$$

$$= \sum_{(\underline{x}_1, \underline{x}_2, \underline{y}) \in T_\delta^n(X_1, X_2, Y)} p(\underline{x}_1) \cdot p(\underline{x}_2, \underline{y}) \tag{39}$$

$$\leq 2^{n(H(X_1,X_2,Y)+\delta)} 2^{-n(H(X_1)-\delta)} \cdot 2^{-n(H(X_2,Y)-\delta)} \tag{40}$$

$$= 2^{-n(I(X_1;X_2,Y)-3\delta)} \tag{41}$$

$$= 2^{-n(I(X_1;Y|X_2)+I(X_1;X_2)-3\delta)} \tag{42}$$

where (40) follows from the discussion in Section III. Similarly, for $j \neq 1$ we have

$$\mathrm{P}(E_{1j}) \leq 2^{-n(I(X_2;Y|X_1)+I(X_1;X_2)-3\delta)} \tag{43}$$

and for $i \neq 1$, $j \neq 1$ we have

$$\mathrm{P}(E_{ij}) \leq 2^{-n(I(X_1,X_2;Y)+I(X_1;X_2)-4\delta)}. \tag{44}$$

Therefore the error probability is bounded by

$$P_e \leq \mathrm{P}(E_{11}^C) + 2^{nR_1} 2^{-n(I(X_1;Y|X_2)+I(X_1;X_2)-3\delta)} \tag{45}$$

$$+ 2^{nR_2} 2^{-n(I(X_2;Y|X_1)+I(X_1;X_2)-3\delta)} + 2^{n(R_1+R_2)} 2^{-n(I(X_1,X_1;Y)+I(X_1;X_2)-4\delta)}. \tag{46}$$

By the generalization of (25) to three random sequences, we have $\mathrm{P}(E_{11}^C) \to 0$ as $n \to \infty$. By choosing $\delta > 0$ small and letting $n \to \infty$ we can thus achieve low error probability as long as

$$R < R_1 + R_2 - I(X_1; X_2) \tag{47}$$

$$R_1 < I(X_1; Y|X_2) + I(X_1; X_2) \tag{48}$$

$$R_2 < I(X_2; Y|X_1) + I(X_1; X_2) \tag{49}$$

$$R_1 + R_2 < I(X_1, X_2; Y) + I(X_1; X_2) \tag{50}$$

where (47) follows from (34). These constraints translate to the following constraints on R:

$$R < R_1 + R_2 - I(X_1; X_2) \tag{51}$$

$$R < I(X_2; Y|X_1) + R_1 \tag{52}$$

$$R < I(X_1; Y|X_2) + R_2 \tag{53}$$

$$R < I(X_1, X_2; Y). \tag{54}$$

This proves the theorem. ∎

V. General Networks with Multi Access Interference

Recall that for networks of MACs there is one output Y_u associated with every vertex $u \in \mathcal{V}$ and one input $X_{v,u}$ associated with every edge $(v, u) \in \mathcal{E}$. For a fixed vertex u, define $X_{\mathcal{V},u} = \{X_{v,u} : (v, u) \in \mathcal{E}\}$. The MAC corresponding to node u is therefore $p(y_u|x_{\mathcal{V},u})$.

Beyond the finite field model of [1], [2], the unicast and multicast capacities for such networks are not known. Moreover, separating source and channel coding [5] and separating channel and network coding [1] is suboptimal. We seek an achievable rate strictly exceeding the rate for independent

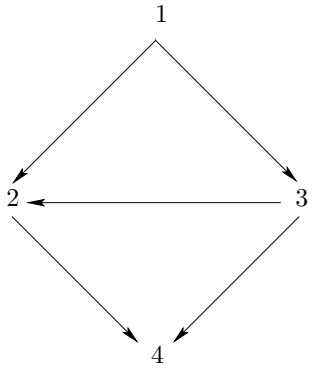


Fig. 2. Every node $i = 1,...,4$ has output Y_i and every edge has input X_{ij}.

inputs. We will consider joint distributions over the inputs of every MAC in the network, but independent inputs across different MACs. Using Theorem 3, we can sometimes achieve rates larger than the rate for independent inputs. Furthermore, the rate has an interpretation as a flow problem with losses incurred at nodes.

A. An example

Consider the network in Fig. 2 with two MACs $p(y_2|x_{12}, x_{32})$ and $p(y_4|x_{24}, x_{34})$ and one point-to-point channel of capacity C_{13}. Applying Theorem 3 to both MACs, we have the following constraints

$$\begin{aligned} R_2 \leq \min(&R_{12} + R_{32} - I(X_{12}, X_{32}), \\ &I(X_{12}; Y_2|X_{32}) + R_{32}, \\ &R_{12} + I(X_{32}; Y_2|X_{12}), \\ &I(X_{12}, X_{32}; Y_2)) \end{aligned} \tag{55}$$

$$\begin{aligned} R_4 \leq \min(&R_{24} + R_{34} - I(X_{24}, X_{34}), \\ &I(X_{24}; Y_4|X_{34}) + R_{34}, \\ &R_{24} + I(X_{34}; Y_4|X_{24}), \\ &I(X_{24}, X_{34}; Y_4)). \end{aligned} \tag{56}$$

We also have flow conservation constraints at nodes 2 and 3

$$R_{24} \leq R_2 \tag{57}$$

$$R_{32} + R_{34} \leq R_3 \leq C_{13} \tag{58}$$

and the entropy constraints

$$R_{ij} \leq H(X_{ij}), \quad \forall (i,j) \in \mathcal{E}. \tag{59}$$

The bounds (55)-(59) define a polytope of achievable rates. The structure of the constraints is reminiscent of a flow formulation. However, the MAC constraints cannot be reduced to flow conservation constraints and therefore our problem is not a "classical" network flow problem. The maximal end-to-end rate from node 1 to 4 is the maximum R_4 over the joint distribution of the inputs X_{ij}.

B. A general formulation

Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of MACs, and suppose for simplicity that every MAC has exactly two inputs and is deterministic. Our analysis extends to less or more than two inputs and noisy MACs, but to streamline the presentation we consider the simpler problem. Applying Theorem 3, for every vertex u with inputs from nodes v_1 and v_2, we have

$$\begin{aligned} R_u \leq \min(&R_{v_1,u} + R_{v_2,u} - I(X_{v_1,u}; X_{v_2,u}), \\ &H(Y|X_{v_2,u}) + R_{v_2,u}, \\ &R_{v_1,u} + H(Y|X_{v_1,u}), \\ &H(Y)). \end{aligned} \tag{60}$$

Furthermore, no node can transmit at a higher rate than the rate it receives, so we have

$$\sum_{v:(u,v)\in\mathcal{E}} R_{u,v} \leq R_u. \tag{61}$$

We also have the capacity constraints

$$R_{u,v} \leq H(X_{u,v}) \tag{62}$$

$$R_u \leq H(Y_u). \tag{63}$$

The bounds (60)-(63) define a polytope of achievable rates.

C. A network flow formulation with node losses

To uncover more clearly the flow structure of our approach, note that the right-hand side of (60) is less than or equal to $R_{v_1,u} + R_{v_2,u}$ and we have

$$R_u \leq R_{v_1,u} + R_{v_2,u} - l_u(R_{v_1,u}, R_{v_2,u}), \tag{64}$$

where we define

$$\begin{aligned} l_u(R_{v_1,u}, R_{v_2,u}) = &R_{v_1,u} + R_{v_2,u} - \\ &\min(R_{v_1,u} + R_{v_2,u} - I(X_{v_1,u}; X_{v_2,u}), \\ &H(Y|X_{v_2,u}) + R_{v_2,u}, \\ &R_{v_1,u} + H(Y|X_{v_1,u}), \\ &H(Y)). \end{aligned} \tag{65}$$

The quantity $l_u(R_{v_1,u}, R_{v_2,u})$ can be thought of as a loss of rate at node u. Combining (61) and (60), we have

$$\sum_{v:(u,v)\in\mathcal{E}} R_{u,v} \leq \left[\sum_{v:(v,u)\in\mathcal{E}} R_{v,u} \right] - l_u. \tag{66}$$

We also have the edge constraints

$$R_{u,v} \leq H(X_{u,v}), \quad \forall (u,v) \in \mathcal{E}. \tag{67}$$

We thus have flow conservation constraints on a capacitated graph with a loss l_u incurred at node u. The problem of maximizing the unicast rate from s to t becomes

$$\text{maximize} \quad R$$

subject to

$$\sum_{v:(u,v)\in\mathcal{E}} R_{u,v} - \sum_{v:(v,u)\in\mathcal{E}} R_{v,u} = \begin{cases} R & u = s, \\ -R - l_t & u = t, \\ -l_u & \text{else}, \end{cases} \quad (68)$$

$$R_{u,v} \leq H(X_{u,v}), \quad \forall (u,v) \in \mathcal{E}, \quad (69)$$

$$l_u = f(R_{v_1,u}, R_{v_2,u}). \quad (70)$$

Note that l_u depends on $p(x_{v_1,u}, x_{v_2,u})$. Furthermore, it suffices to optimize over joint input distributions that factor as

$$p(\underline{x}) = p(x_{\mathcal{V},u_1}) \cdot ... \cdot (x_{\mathcal{V},u_{|\mathcal{V}|}}). \quad (71)$$

VI. Concluding Remarks

We developed a coding scheme for general MACs that introduces correlation between MAC inputs and thereby improves on techniques restricted to independent inputs. The method was extended to apply to networks of MACs using a flow formulation. In general, our technique does not achieve the cut-set bound. A promising direction of research might be to investigate the utility of dependence balance [13] concepts for our setup.

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