Systematic Design of Unitary Space-Time Constellations

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Abstract

We propose a systematic method for creating constellations of unitary space-time signals for multiple-antenna communication links. Unitary space-time signals, which are orthonormal in time across the antennas, have been shown to be well-tailored to a Rayleigh fading channel where neither the transmitter nor the receiver knows the fading coefficients. The signals can achieve low probability of error by exploiting multiple antenna diversity at the transmitter.

Our construction begins with the first signal in the constellation—an oblong complex-valued matrix whose columns are orthonormal—and systematically produces the remaining signals by succesively rotating this signal in a high-dimensional complex space. This construction easily produces large constellations of high-dimensional signals. We demonstrate its efficacy through examples involving one, two, and three transmitter antennas.

Index Terms—Multi-element antenna arrays, wireless communications, fading channels, transmit diversity, receive diversity

1 Introduction

Recent theoretical treatments have shown that communication systems that employ multiple antennas can have very high channel capacities, especially in Rayleigh flat-fading environments [3], [10], [5]. In [3], a constructive approach to achieving some of this capacity is proposed, requiring potentially sophisticated signal processing in space and time (space-time) under the assumption that the receiver knows the complex-valued Rayleigh fading coefficients. Under the same assumption, [8] presents a trellis-based approach for designing space-time codes, and [9] gives a space-time signaling method based on orthogonal designs. However, the known-channel assumption may not be realistic in a rapidly changing fading environment or with a large number of transmitter antennas.

A new class of *unitary space-time* signals is proposed in [6] that are well-taylored for flat-fading channels where neither the transmitter nor the receiver knows the fading coefficients. Suppose there are M transmitter antennas, and that we transmit signals in blocks of T time samples. Then a constellation of L unitary space-time signals $S_\ell = \sqrt{T}\Phi_\ell$, $\ell = 1, \ldots, L$, has the defining property that $\Phi_1, \ldots, \Phi_L$ are $T \times M$ complex-valued matrices obeying $\Phi_1^\dagger \Phi_1 = \ldots = \Phi_L^\dagger \Phi_L = I$. Of necessity, $M \leq T$. The mth column of any S_ℓ contains the signal transmitted on antenna m as a function of time. Essentially, the directions, and not the lengths, of the orthonormal vectors in Φ_ℓ carry the message information.

Intuitive and theoretical arguments in [5] and [6] suggest that unitary space-time signals are not only simple to decode, but they also attain capacity when used in conjunction with coding in a multiple-antenna Rayleigh fading channel when either $T \gg M$ or the signal-to-noise ratio is reasonably large and $M < T$. Hence, there is a strong motivation for designing good unitary space-time constellations. Some successful unitary space-time constellations are designed and demonstrated in [6] but the techniques used therein can not be readily extended to large constellations or to signals of high dimension. This paper devises some simple algorithms for designing effective unitary space-time constellations.

The only *a priori* structure on a unitary space-time constellation is the time-orthonormality of the signals. Constellation design is viewed in [6] as a difficult and cumbersome search and optimization problem. But, as we show, by imposing additional structure on these signals and asking that their generation be systematic, we can construct and optimize a constellation with relatively little effort. We present the design in two disparate but ultimately equivalent ways. The

first approach, Section 3, is Fourier-based and uses ideas from signal processing theory. The second approach, Section 4, is algebraic and uses ideas from coding theory. Section 5 demonstrates the performance of these approaches on a multiple-antenna Rayleigh fading channel where neither the receiver nor the transmitter knows the propagation coefficients. The performances of constellations for use with one, two, and three transmitter antennas are compared.

Throughout the paper, we concentrate on modulation and constellation design, and do not address coding issues that lower error probability by adding redundancy. We focus, instead, on raw or uncoded signal and bit error probabilities. Because these constellations are justified on an information-theoretic basis for multiple-antenna Rayleigh fading channels, they should be well-suited for coding and achieving channel capacity on these channels.

The next section introduces the model for the channel. The following notation is used throughout the paper: Two complex vectors, a and b, are *orthogonal* if $a^\dagger b = 0$, where the superscript † denotes "conjugate transpose." The zero-mean, unit-variance, circularly-symmetric, complex Gaussian distribution is denoted by $\mathcal{C}N(0,1)$.

2 Rayleigh Flat-Fading Channel Model

Consider a communication link comprising M transmitter antennas and N receiver antennas that operates in a Rayleigh flat-fading environment. Each receiver antenna responds to each transmitter antenna through a statistically independent fading coefficient that is constant for T symbol periods. The received signals are corrupted by additive noise that is statistically independent among the N receivers and the T symbol periods. We use complex baseband notation: during the T-symbol interval, we transmit the signal $\{s_{tm},\ t = 1,\ldots,T,\ m = 1,\ldots,M\}$ on M antennas, and we receive the noisy signal $\{x_{tn},\ t = 1,\ldots,T,\ n = 1,\ldots,N\}$ on N receivers,

$$x_{tn} = \sqrt{\rho/M}\sum_{m=1}^{M} h_{mn}s_{tm} + w_{tn}, \qquad t = 1,\ldots T,\ \ n = 1\ldots N. \tag{1}$$

Here h_{mn} is the complex-valued fading coefficient between the mth transmitter antenna and the nth receiver antenna. The fading coefficients are constant for $t = 1,\ldots T$, and they are independent with respect to m and n and $\mathcal{C}N(0,1)$ distributed. The additive noise at time t and receiver antenna n is denoted w_{tn}, and is independent (with respect to both t and n), identically distributed $\mathcal{C}N(0,1)$.

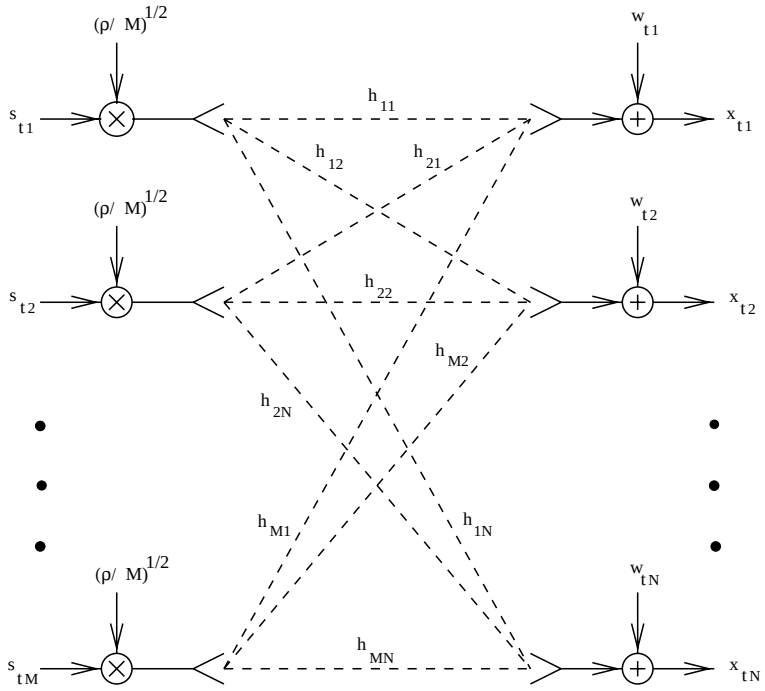


Figure 1: Wireless link comprising M transmitter and N receiver antennas. Every receiver antenna is connected to every transmitter antenna through an independent, random, unknown propagation coefficient having Rayleigh distributed magnitude and uniformly distributed phase. Normalization ensures that the total expected transmitted power is independent of M for a fixed ρ.

The quantities in the signal model (1) are normalized so that ρ represents the expected signal-to-noise ratio (SNR) at each receiver antenna, independently of M. We assume that the realizations of h_{mn}, $m = 1, \ldots, M$, $n = 1, \ldots, N$ are not known to the receiver or transmitter. See Figure 1 and [5] for more details.

We assume that the fading coefficients change to new independent realizations every $T \geq 1$ symbol periods. This piecewise constant fading process mimics, in a tractable manner, the approximate coherence interval of a continuously fading process. Furthermore, it is a very accurate representation of many TDMA, frequency hopping, or block-interleaved systems. Each channel use (consisting of a block of T transmitted symbols) is independent of every other.

Equation (1) can be written compactly as

$$X = \sqrt{\frac{\rho}{M}} SH + W \tag{2}$$

where X is the $T \times N$ complex matrix of received signals, S is the $T \times M$ matrix of transmitted signals, H is the $M \times N$ matrix of Rayleigh fading coefficients, and W is the $T \times N$ matrix of additive receiver noise. In this notation, the M columns of S represent the signals sent on the M transmitter antennas as functions of time.

We will use a constellation of unitary space-time signals $S_1 = \sqrt{T}\Phi_1, \ldots, S_L = \sqrt{T}\Phi_L$ to transmit binary information over this link. It is shown in [5] and [6] that the capacity-achieving distribution for reasonably large T or ρ is $S = \sqrt{T}\Phi$, where $\Phi^\dagger\Phi = I$ and Φ is isotropically distributed. Details about this distribution may be found in [5], but it suffices to say that its defining characteristic is that Φ and $\Theta\Phi$ have the same distribution for any deterministic unitary Θ.

It is also shown in [6] that the maximum likelihood decoder for a constellation of unitary space-time signals is

$$\Phi_{\text{ml}} = \arg \max_{\Phi_\ell = \Phi_1, \ldots, \Phi_L} \text{tr}\,\{X^\dagger \Phi_\ell \Phi_\ell^\dagger X\},$$

and that the two-signal probability of mistaking S_1 for S_2 or vice-versa is

$$P_e = \sum_j \text{Res}_{\omega = ia_j} \left\{ -\frac{1}{\omega + i/2} \prod_{\substack{m=1 \\ d_m < 1}}^{M} \left[\frac{1 + \rho T/M}{(\rho T/M)^2 (1 - d_m^2)(\omega^2 + a_m^2)} \right]^N \right\}, \tag{3}$$

where $1 \geq d_1 \geq \ldots \geq d_M \geq 0$ are the singular values of the $M \times M$ correlation matrix $\Phi_2^\dagger \Phi_1$, and

$$a_m \stackrel{\text{def}}{=} \sqrt{\frac{1}{4} + \frac{1 + \rho T/M}{(\rho T/M)^2 (1 - d_m^2)}}.$$

The probability of error P_e decreases as any d_m decreases, and has Chernoff upper bound

$$P_e \leq \frac{1}{2} \prod_{m=1}^{M} \left[\frac{1}{1 + \frac{(\rho T/M)^2 (1 - d_m^2)}{4(1 + \rho T/M)}} \right]^N. \tag{4}$$

The probability of error is lowest when $d_1 = \ldots = d_M = 0$ and highest when $d_1 = \ldots = d_M = 1$. We obtain $d_1 = \ldots = d_M = 0$ when the columns of S_2 are all orthogonal to all the columns of S_1. The ideal constellation $S_1, \ldots, S_L$ therefore has all the columns of S_ℓ orthogonal to all the columns of $S_{\ell'}$ for $\ell' \neq \ell = 1, \ldots, L$. However, because the columns of each S_ℓ are within themselves orthogonal to one another, all the pairwise $d_1, \ldots, d_M$ cannot all be made zero if $L > T/M$.

Let

$$\left\| \Phi_{\ell'}^{\dagger} \Phi_{\ell} \right\| = \sqrt{\frac{d_1^2 + \ldots + d_M^2}{M}} \tag{5}$$

denote the root-mean-square norm of the singular values of the correlation matrix (this is a scaled Frobenius norm). This norm lies between zero and one, with small norm generally implying small probability of error. For a given constellation, let

$$\delta = \max_{1 \le \ell < \ell' \le L} \left\| \Phi_{\ell}^{\dagger} \Phi_{\ell'} \right\| . \tag{6}$$

In general, we will be content with choosing constellations that have δ as small as possible.

This design criterion is markedly different from the familiar maximum-Euclidean-distance criterion, and it arises entirely because the fading coefficients are unknown to the receiver. Because of this, antipodal pairs of signals $\pm S$ are indistinguishable, for example. In constructing a constellation of unitary space-time signals, we note that the probability of error is invariant to two types of transformations: 1) left multiplication by a common $T \times T$ unitary matrix, $\Phi_\ell \rightarrow \Psi^{\dagger} \Phi_\ell$, $\ell = 1, \cdots, L$; 2) right multiplication by individual $M \times M$ unitary matrices, $\Phi_\ell \rightarrow \Phi_\ell \Upsilon_\ell$, $\ell = 1, \cdots, L$; see [6].

To transmit R bits per channel use, we need a constellation of at least $L = 2^{RT}$ signals. For example, if $R = 2$ bits/channel use and $T = 10$, then $L = 2^{20} \approx 10^6$. Generating and storing this many $M \times T$ complex matrices is cumbersome if the signals are not provided with some additional structure. Furthermore, it is not obvious how to ensure that the generated signals have low probability of error. In the next section, we describe a systematic approach to create signals with low probability of error and that requires storage of only Φ_1 and a $T \times T$ diagonal matrix with which to generate $\Phi_2, \ldots, \Phi_L$.

3 Fourier-based Construction

In this section we present a Fourier-based construction of a constellation of unitary space-time signals. Section 3.1 gives the intuition behind the construction, which has a block-circulant signal correlation structure. Section 3.2 then proves that this construction yields all constellations having a block-circulant correlation structure.

3.1 Fourier-based construction has block-circulant correlation

We begin with $M = 1$ transmitter antenna; we therefore need L unit vectors in a T dimensional complex space where, in general, $L \gg T$. Clearly these vectors will form an overcomplete or linear dependent system. Overcomplete representations are becoming increasingly popular in signal representation and are often studied using the mathematical technique of *frames* [2]. We say that a collection of L unit column vectors Φ_ℓ in a T dimensional space form a *tight frame* if all of the eigenvalues of the $T \times T$ matrix $\sum_{\ell=1}^{L} \Phi_\ell \Phi_\ell^\dagger$ are equal, implying that

$$\sum_{\ell=1}^{L} \Phi_\ell \Phi_\ell^\dagger = \frac{L}{T} \cdot I.$$

While the details of frame theory go beyond the scope of this paper, we use a well-known result that any tight frame with L vectors in T dimensions can be seen as the projection into a T dimensional space of an orthonormal basis in L dimensions, see, e.g., [4]. Balan and Daubechies construct tight frames of unit vectors by projecting an L dimensional Discrete Fourier Transform (DFT) basis onto a T dimensional space [1]. The projection simply retains the first T components of the L dimensional vectors. Inspired by this construction, we propose the one antenna constellation

$$\Phi_\ell = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{L}(\ell-1)} \\ e^{i\frac{2\pi}{L}2(\ell-1)} \\ \vdots \\ e^{i\frac{2\pi}{L}(T-1)(\ell-1)} \end{bmatrix}. \tag{7}$$

For this choice, we obtain

$$d_1 = \left|\Phi_\ell^\dagger \Phi_{\ell'}\right| = \begin{cases} 1 & (\ell' = \ell) \\ \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}(t-1)(\ell'-\ell)}\right| = \left|\frac{\sin\left(\pi(\ell'-\ell)T/L\right)}{T\sin\left(\pi(\ell'-\ell)/L\right)}\right| & (\ell' \neq \ell). \end{cases}$$

As shown in (3) and (4), the two-signal probability of error depends only on the correlation d_1 and decreases as d_1 decreases. We observe that:

1. The correlation between Φ_ℓ and $\Phi_{\ell'}$ depends only on $(\ell'-\ell)$ mod L; the correlation structure of

the entire constellation is therefore circulant and it suffices to consider $\left|\Phi_1^\dagger \Phi_\ell\right|$ for $\ell = 2, \ldots, L$.

2. The correlation structure behaves roughly like a sinc function, and hence equation (6) yields $\delta = |\Phi_\ell^\dagger \Phi_{\ell+1}| = 1 - O(1/L^2)$ as $L \to \infty$. For large L, equation (3) (with $M = 1$ and $d_1 = |\Phi_\ell^\dagger \Phi_{\ell+1}|$) therefore implies that the probability of mistaking Φ_ℓ for its immediate neighbors is high; this is decidedly undesirable. Figure 2 shows the correlation structure for $T = 6$ and $L = 64$, for which $\delta = 0.986$.

3. If T divides L, the vectors Φ_ℓ and $\Phi_{(\ell+L/T) \bmod L}$ are orthogonal. This property can be used to construct multiple antenna codes as shown in the Appendix. However, these constellation constructions are not pursued in the rest of this paper.

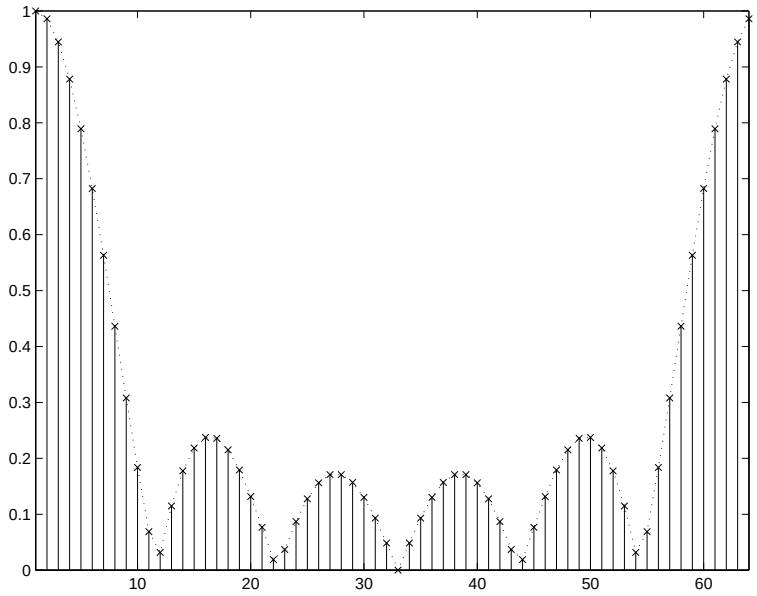


Figure 2: Correlation structure of signals in equation (7) when $T = 6$ and $L = 64$, which implies a transmission rate of $R = 1$ bit per channel use. We clearly see the sinc-like behavior. The maximum correlation δ as defined in equation (6) (which is achieved with Φ_ℓ and $\Phi_{\ell+1}$) is 0.986.

Property 2 suggests that $\Phi_1, \ldots, \Phi_L$ given by (7) are a poor choice of signals, especially if L is large. However, we are not necessarily constrained to choose the first T rows of the $L \times L$ DFT matrix as is done in (7). To lower the correlation between neighbors, we may consider choosing

another set of T components. We thus let

$$\Phi_\ell = \frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\frac{2\pi}{L}u_1(\ell-1)} \\ e^{i\frac{2\pi}{L}u_2(\ell-1)} \\ \vdots \\ e^{i\frac{2\pi}{L}u_T(\ell-1)} \end{bmatrix}, \tag{8}$$

where, without loss of generality, $0 \leq u_1, \ldots, u_T \leq L-1$. We still have a circulant correlation structure because

$$\left|\Phi_\ell^\dagger \Phi_{\ell'}\right| = \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}u_t(\ell'-\ell)}\right|.$$

We can now choose the "frequencies" $u_1, \ldots, u_T$ to get the lowest possible correlations. As mentioned in Property 1, because of the circulant structure it suffices to look at

$$\left|\Phi_1^\dagger \Phi_\ell\right| = \left|\sum_{t=1}^{T}[\Phi_\ell]_t\right| = \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}u_t(\ell-1)}\right|, \qquad \ell = 2, \ldots, L, \tag{9}$$

where $[\cdot]_t$ denotes the tth component of $[\cdot]$. We wish to find $u_1, \ldots, u_T$ achieving

$$\min_{0\leq u_1,\ldots,u_T\leq L-1} \max_{\ell=2,\ldots,L} \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{L}u_t(\ell-1)}\right| = \min_{0\leq u_1,\ldots,u_T\leq L-1} \delta, \tag{10}$$

where δ (given by (6)) depends on $u_1, \ldots, u_T$. Observe that $\left|\Phi_1^\dagger \Phi_\ell\right|$ can be interpreted as the modulus of the DFT of a length L sequence with the value one at positions $u_1, \ldots, u_T$ and zero elsewhere. Thus one can look at the minimization in (10) as a filter design problem, where the filter is sparse (i.e., only T out of a possible L filter coefficients are nonzero), the response at zero-frequency is unity, and where we choose the locations of the T nonzero coefficients to minimize the response at frequencies that are multiples of $2\pi/L$.

The problem of sparse filter design is analogous to that of aperiodic antenna array design [7]. A conventional linear antenna array having T elements uses half-wavelength spacing between its elements, and it has an angular frequency response having the sinc-like behavior shown in Fig. 2. The width of the central peak at zero-frequency is inversely proportional to the physical length of the array. If one desires the narrower central peak associated with higher angular resolution for the *same* number of elements T, one has to use a longer array. Doubling the spacing to give a uniform

spacing of one wavelength would reduce the width of the central peak by a factor of two, but with the penalty of replicating the angular frequency response at intervals of π (the so-called grating lobe effect). However by using a longer aperiodic array, one can obtain a narrower central peak without introducing grating lobes. Despite much effort, there has never been a completely satisfactory way to design aperiodic arrays: for small arrays one can use exhaustive search, whereas, for large arrays, random search strategies seem to be the only resort. In our optimizations, we therefore also generally employ a random search. Figure 3 shows the results of such a search. Observe how optimizing over $u_1, \ldots, u_T$ allows a much better correlation structure than in Figure 2.

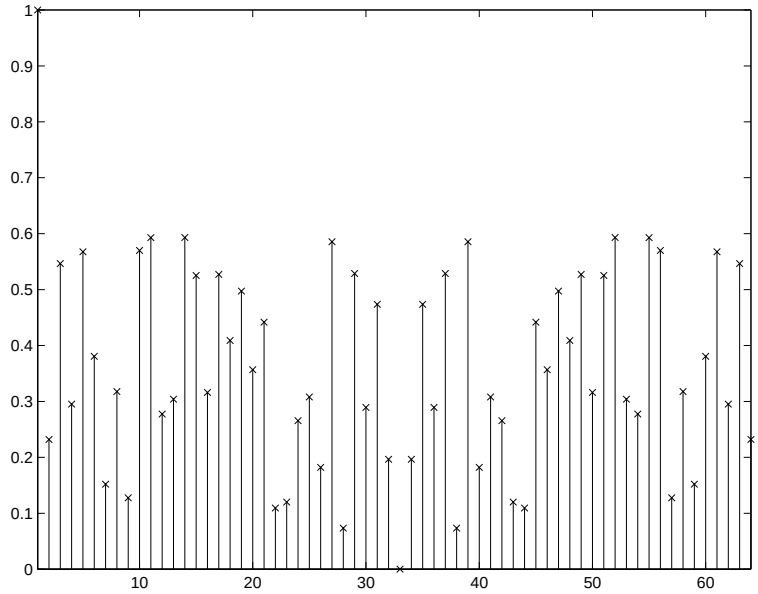


Figure 3: Correlation structure when choosing nearly optimal $u_1, \ldots, u_T$ in (8), with $T = 6$ and $L = 64$. Here $u = [19\ 23\ 24\ 38\ 51\ 58]$. The maximum correlation is $\delta = 0.593$.

We now show how we can generalize this single-antenna construction to $M > 1$ antennas. In the single-antenna case, each signal can be written as

$$\Phi_\ell = \Theta^{\ell-1}\, \Phi_1, \tag{11}$$

where Θ is a $T \times T$ matrix whose diagonal elements are $e^{i2\pi u_1/L}, \ldots, e^{i2\pi u_T/L}$ and Φ_1 is $1/\sqrt{T}$ times a vector of all ones. Note that Θ is a unitary matrix and that $\Theta^L = I_L$. Geometrically, the construction can be interpreted as rotating an initial vector through T-dimensional complex space

using a matrix which is the Lth root of unity. The matrix is chosen so that the resulting vectors have as little correlation as possible, and after L rotations the vector is brought back to its initial position.

For $M > 1$ transmitter antennas, let Φ_1 be a $T \times M$ matrix with $\Phi_1^\dagger \Phi_1 = I_M$ and form the constellation again by applying (11). Because Θ is an Lth root of unity, we have a block-circulant structure in the sense that the $M \times M$ matrix $\Phi_\ell^\dagger \Phi_{\ell'}$ only depends on $(\ell' - \ell) \bmod L$, and because Θ is unitary, $\Phi_\ell^\dagger \Phi_\ell = I_M$. Geometrically, this construction can be interpreted as rotating an initial M-dimensional subspace using an Lth root of unity to form L different M-dimensional subspaces.

As noted in equation (6), a constellation with small probability of error generally has small δ. We may therefore choose $u_1, \ldots, u_T$ to achieve

$$\min_{0 \le u_1, \ldots, u_T \le L-1} \delta \;=\; \min_{0 \le u_1, \ldots, u_T \le L-1} \max_{\ell = 2, \ldots, L} \left\| \Phi_1^\dagger \Phi_\ell \right\|. \tag{12}$$

A simple method to build a starting matrix Φ_1 is to choose M distinct columns of a $T \times T$ DFT matrix. This ensures that $\Phi_1^\dagger \Phi_1 = I_M$.

In the next section we show that the above construction generates all constellations with circulant correlation structure.

Remark: The starting unit vectors Φ_1 that we have used so far—either $1/\sqrt{T}$ times a vector of all ones, or the columns of a DFT matrix—have components all with modulus $1/\sqrt{T}$. There is no particular need to impose this constraint, and experiments indicate that optimizations that allow the moduli of the starting vector components to vary (but maintain unit norms for the columns of each Φ_ℓ) can yield even smaller values of δ. For simplicity, we do not pursue these optimizations.

3.2 Block-circulant correlation has Fourier-based construction

In the previous section we propose a certain constellation and observe that it leads to a circulant correlation structure. In this section we take the converse approach and characterize all constellations which yield a circulant correlation structure.

Let $\{\Phi_1, \cdots, \Phi_L\}$ be some constellation of unitary space-time signals. We impose the block-circulant correlation structure,

$$\Phi_\ell^\dagger \Phi_{\ell'} = F_{(\ell' - \ell) \bmod L}, \quad \ell = 1, \ldots, L, \quad \ell' = 1, \ldots, L, \tag{13}$$

where F_ℓ are $M \times M$ matrices and the orthonormality of the columns of each signal implies that $F_0 = I_M$. It is also easy to see that

$$F_{-\ell} = F_{L-\ell} = \Phi_\ell^\dagger \Phi_L = F_{\ell-L}^\dagger = F_\ell^\dagger. \tag{14}$$

The block-circulant correlation structure implies that the exact conditional probability of error for deciding which of the L signals was transmitted is the same for all L signals.

We now take the double (i.e., both in ℓ and ℓ') discrete Fourier transform of both sides of (13) to obtain

$$\sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} \Phi_\ell^\dagger \Phi_{\ell'} e^{-i\frac{2\pi}{L}[(\ell'-1)(n'-1)-(\ell-1)(n-1)]} = \widehat{\Phi}_n^\dagger \widehat{\Phi}_{n'} = L\widehat{F}_n \delta_{(n'-n) \bmod L}\ , \tag{15}$$

where Fourier transforms, which are matrix valued, are denoted by the hatted quantities,

$$\widehat{\Phi}_n = \sum_{\ell=1}^{L} \Phi_\ell e^{-i\frac{2\pi}{L}(\ell-1)(n-1)} \quad \text{and} \quad \widehat{F}_n = \sum_{\ell=1}^{L} F_\ell e^{-i\frac{2\pi}{L}(\ell-1)(n-1)}. \tag{16}$$

Equation (15) is equivalent to the well-known result that a circulant matrix is diagonal in the Fourier domain.

According to (15), the L Fourier coefficients $\widehat{\Phi}_n$, each a $T \times M$ matrix, are mutually orthogonal. Consequently all but at most T of the coefficient matrices are zero. We denote the T possibly nonzero Fourier coefficient matrices by $\widehat{\Phi}_{u_1}, \cdots, \widehat{\Phi}_{u_T}$ where $0 \le u_1, \ldots, u_T \le L-1$. The signals are thus given by the inverse Fourier transform,

$$\Phi_\ell = \frac{1}{L} \sum_{t=1}^{T} \widehat{\Phi}_{u_t} e^{i\frac{2\pi}{L}u_t(\ell-1)}, \qquad \ell = 1, \cdots, L. \tag{17}$$

When exactly T coefficient matrices are nonzero, then orthogonality requires them all to be rank one, for there cannot be more than T linearly independent T-dimensional vectors. When only $T-1$ coefficients are nonzero, at most one of them can have rank two while the others have rank one. The rank-two matrix can always be written as the sum of two rank-one matrices; for example, take its singular value decomposition and write the two-element diagonal matrix as a sum of two one-element diagonal matrices. Then we again have a sum (17) with T terms where each coefficient matrix has rank 1; the only difference is that the two coefficient matrices coming from the split have the same frequency term $e^{i\frac{2\pi}{L}u_t(\ell-1)}$. Similar arguments for $T-2$ or fewer nonzero coefficients

yield the same conclusion that all coefficient matrices in (17) can be made to have rank one.

Next consider the $T \times T$ matrix formed by taking the first column of each $\widehat{\Phi}_{u_t}$, $t = 1, \ldots, T$. The columns of this matrix are then orthogonal, but not necessarily orthonormal. Thus, this matrix can be written as ΨD, where Ψ is a $T \times T$ unitary matrix and D is diagonal. Now $\Psi^\dagger$ times the first column of $\widehat{\Phi}_{u_t}$ is a vector with only the t-th component nonzero. Because $\widehat{\Phi}_{u_t}$ is rank one, all its columns are scaled copies of one another. Hence $\Psi^\dagger \widehat{\Phi}_{u_t}$ is a matrix with only its t-th row nonzero. Recall that the error performance of a constellation does not change when applying the transformation

$$\Phi_\ell \mapsto \Psi^\dagger \Phi_\ell, \quad \ell = 1, \ldots, L, \tag{18}$$

for unitary Ψ. From (17) we see that this transformation is equivalently applied to the Fourier coefficient matrices: $\widehat{\Phi}_{u_t} \mapsto \Psi^\dagger \widehat{\Phi}_{u_t}$, $t = 1, \ldots, T$. After this transformation, $\widehat{\Phi}_{u_t}$ is zero except in its t-th row. The set of Fourier coefficients are therefore orthogonal by virtue of their disjoint row support.

The signal Φ_1 combines the different nonzero rows of the $\widehat{\Phi}_{u_t}$ matrices,

$$\Phi_1 = \frac{1}{L} \sum_{t=1}^{T} \widehat{\Phi}_{u_t}.$$

Any other signal Φ_ℓ can be formed from Φ_1 by multiplying the t-th row by $e^{i\frac{2\pi}{L}u_t(\ell-1)}$ as in (17). Hence, Φ_ℓ can be expressed more conveniently as a $T \times T$ diagonal unitary matrix Θ that is raised to the $(\ell - 1)$th power, times the $T \times M$ matrix Φ_1,

$$\Phi_\ell = \Theta^{\ell-1} \Phi_1, \tag{19}$$

where

$$\Theta = \begin{bmatrix} e^{i\frac{2\pi}{L}u_1} & 0 & \cdots \\ & \ddots & \\ 0 & \cdots & e^{i\frac{2\pi}{L}u_T} \end{bmatrix}, \qquad 0 \le u_1, \ldots, u_T \le L - 1. \tag{20}$$

Since Φ_1 only underwent the unitary transformation (18), it still has the property that $\Phi_1^\dagger \Phi_1 = I_M$. By (19), the correlation matrix between any two signals has the block-circulant structure (13)

$$\Phi_\ell^\dagger \Phi_{\ell'} = \Phi_1^\dagger \Theta^{\ell'-\ell} \Phi_1. \tag{21}$$

We conclude that any unitary space-time constellation whose correlation matrix is block-circulant can be designed using the methods of Section 3.1. We therefore have the following theorem.

Theorem 1 *Any unitary space-time signal constellation of $T \times M$ matrices $\Phi_1, \ldots, \Phi_L$ with a block-circulant correlation structure is equivalent to one that can be written*

$$\Phi_\ell = \Theta^{\ell-1}\Phi_1, \tag{22}$$

where Φ_1 is a $T \times M$ matrix obeying $\Phi_1^\dagger \Phi_1 = I_M$, and Θ is a $T \times T$ diagonal matrix whose diagonal elements are Lth roots of unity. Conversely, every constellation of the form (22), has a block-circulant correlation structure.

3.3 Multiple index block-circulant structures

The previous constellation construction may be extended to a doubly-indexed construction, $\{\Phi_{\ell_1\ell_2}, \ell_1 = 1, \cdots, L_1;\ \ell_2 = 1, \cdots, L_2\}$, where $L_1 \cdot L_2 = L$, and where the constellation has the following correlation structure,

$$\Phi^\dagger_{\ell_1\ell_2}\Phi_{\ell_1'\ell_2'} = F_{(\ell_1'-\ell_1) \bmod L_1 (\ell_2'-\ell_2) \bmod L_2}.$$

It can be shown this construction yields a constellation that is generated by means of a separate rotation for each index,

$$\Phi_{\ell_1\ell_2} = \Theta_1^{\ell_1-1}\Theta_2^{\ell_2-1}\Phi_1, \quad \ell_1 = 1, \ldots, L_1;\ \ell_2 = 1, \ldots, L_2, \tag{23}$$

where Θ_1 and Θ_2 are diagonal unitary matrices that are L_1th and L_2th roots of I_T, respectively. This construction involves choosing the diagonal elements of Θ_1 and Θ_2, which we label $0 \leq\ u_{11}, \ldots, u_{1T} \leq L_1 - 1$ and $0 \leq u_{21}, \ldots, u_{2T} \leq L_2 - 1$. The constellation is therefore completely determined by the $T \times M$ matrix Φ_1, and the $2 \times T$ matrix U whose entries are u_{kt}, $k = 1, 2$, $t = 1, \ldots, T$.

This construction extends readily to a K-indexed constellation in which $L = \prod_{k=1}^K L_k$ and U is a $K \times T$ matrix.

4 Equivalent Algebraic Construction

The constellation construction described in the previous section can also be viewed algebraically, and in this section we create a constellation of signals by mapping a linear block code into complex signal matrices. The code is over the ring of integers modulo q and the number of codewords is equal to the number of desired signals L. We will relate q to L shortly, and we begin by describing the construction for $M = 1$ transmitter antenna.

Let $\mathsf{R}_q = \{0, \ldots, q-1\}$ be the ring of integers modulo q, and let $\mathcal{C} = \{c_1, \cdots, c_L\}$ denote a linear code over R_q of length T and containing L codewords. Each element c_ℓ of $\mathcal{C}$ is a vector of T integers in $\{0, \ldots, q-1\}$. Because the code is linear it contains the all-zero vector, and if c_ℓ and $c_{\ell'}$ are in $\mathcal{C}$ then so is $ac_\ell + bc_{\ell'}$ for any $a, b \in \mathsf{R}_q$.

We map these codewords into signals by mapping the T integers in a codeword to the T components of a complex signal using the function

$$\phi(j) = \frac{1}{\sqrt{T}} e^{i\frac{2\pi}{q}j}, \qquad j = 0, \ldots, q-1.$$

Note that addition modulo q for the argument corresponds to complex multiplication for the function value. By letting the ϕ function work on vectors, we effectively obtain the one-antenna constellation

$$\Phi_\ell = \phi(c_\ell) = \frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\frac{2\pi}{q}[c_\ell]_1} \\ e^{i\frac{2\pi}{q}[c_\ell]_2} \\ \vdots \\ e^{i\frac{2\pi}{q}[c_\ell]_T} \end{bmatrix}, \qquad 1 \le \ell \le L.$$

Let c_1 be the all zero codeword; then Φ_1 is $1/\sqrt{T}$ times a vector of all ones. We show that the maximum correlation of the resulting constellation is given by

$$\max_{\ell=2,\ldots,L} \left| \sum_{t=1}^{T} [\Phi_\ell]_t \right|,$$

where $[\cdot]_t$ again denotes the tth component of $[\cdot]$ (and the arithmetic is in the field of complex numbers). To see this, pick two different signals Φ_ℓ and $\Phi_{\ell'}$. By definition, $\Phi_\ell = \phi(c_\ell)$ and

$\Phi_{\ell'} = \phi(c_{\ell'})$ for some $c_\ell, c_{\ell'} \in \mathcal{C}$. Thus,

$$\begin{aligned}\Phi_\ell^\dagger \Phi_{\ell'} &= \sum_{t=1}^{T} \phi^*([c_\ell]_t)\phi([c_{\ell'}]_t) = \frac{1}{T}\sum_{t=1}^{T} e^{-i\frac{2\pi}{q}[c_\ell]_t} e^{i\frac{2\pi}{q}[c_{\ell'}]_t} \\ &= \frac{1}{T}\sum_{t=1}^{T} e^{i\frac{2\pi}{q}[c_{\ell'}-c_\ell]_t} = \frac{1}{T}\sum_{t=1}^{T} e^{i\frac{2\pi}{q}[c_{\ell''}]_t}\end{aligned}$$

for some ℓ'', where the last equality follows from the code's linearity. Therefore, as in (9), in searching for constellations that minimize their maximum correlation, we need to check only $L-1$ quantities.

So far, the codes $\mathcal{C}$ are restricted to be linear but are otherwise arbitrary. We further restrict our search by considering codes that have a $K \times T$ generator matrix U of elements in R_q, where K can be thought of as the dimension of the code. The code $\mathcal{C}$ represented by U is the linear span of the rows of U, i.e., every code word can be written in the form

$$c_\ell = \ell \cdot U,$$

for some $1 \times K$ vector $\ell = [\ell_1 \ \cdots \ \ell_K]$ whose K elements are all in R_q. We incorporate this restriction explicitly, because, unlike linear codes over finite fields, linear block codes over R_q do not necessarily have a generator matrix. It follows that the size of the constellation is $L = q^K$.

We may now call ℓ the multi-index (K-index) of the codewords of $\mathcal{C}$. Then the signals have a multi-index circulant correlation structure since

$$\begin{aligned}c_{\ell'} - c_\ell = ([\ell'_1 \ \cdots \ \ell'_K] - [\ell_1 \ \cdots \ \ell_K])\, U &= [\ell'_1 - \ell_1 \ \cdots \ \ell'_K - \ell_K]\, U \\ &= [\ell''_1 \ \cdots \ \ell'_K]\, U \\ &= c_{\ell''}\end{aligned}$$

where all arithmetic is modulo q.

The connection to the constellation construction discussed in Section 3.1 becomes more apparent if we rewrite the codes in the following form. Given U, we let $\Theta_1, \Theta_2, ..., \Theta_K$ be diagonal $T \times T$ complex matrices with entries $[\Theta_k]_{tt} = \phi(U_{kt})$, $k = 1, \ldots, K$, $t = 1, \ldots, T$. Note that $\Theta_1^q = \ldots = \Theta_K^q = I$. The one-antenna constellation determined by the matrix U is then the set of all vectors

of the form

$$\Theta_1^{\ell_1}\,\Theta_2^{\ell_2}\,\cdots\,\Theta_K^{\ell_K}\,\Phi_1.$$

For $K = 1$ and $K = 2$ these are exactly the forms suggested in (19) and (23). Thus, the one-antenna constellation is the image of Φ_1 under the action of the discrete group generated by $\Theta_1, \ldots, \Theta_K$. We can extend this construction to admit multiple-antenna constellations by replacing the vector Φ_1 with a representation of a subspace of larger dimension in exactly the same way as is done in Section 3.1.

To simplify the search for constellations of signals that have low correlation, we restrict U to have be a systematic generator matrix of the form

$$U = [\, I \quad U' \,],$$

where I is the $K \times K$ identity matrix and U' is a $K \times (T-K)$ parity matrix with elements in R_q. Tables 1 and 2 list the best $M = 1$ and $M = 2$ antenna constellations for $T = 8$ we have found in our search. For each constellation, the maximum scaled Frobenius norm δ is given, as described in Section 2 in equations (5) and (6). The constellations all have a systematic representation and the rows of the parity matrix U' are listed. Hence, for a code of dimension K, K rows of $T-K$ elements in $\mathsf{R_q}$ are listed. The starting vector Φ_1 for $M = 1$ is $1/\sqrt{T}$ times a vector of all ones, and the starting matrix Φ_1 for $M = 2$ is $1/\sqrt{T}$ times a matrix whose first column is all ones, and whose second column is $[1\ e^{i\frac{2\pi}{T}}\ \cdots\ e^{i\frac{2\pi}{T}(T-1)}]$.

5 Application to Rayleigh Flat-Fading Channel

We now examine the performance of constellations designed using the methods of Sections 3 and 4 on the multiple-antenna Rayleigh fading channel given in Section 2. We look specifically at $M = 1$, 2, and 3 transmitter antennas, but consider only one receiver antenna because adding more receiver antennas lowers decoding error probability but does not fundamentally affect signal constellation design. For simplicity, we consider $R = 1$ bit/channel use and we assume that the fading coefficients are constant for $T = 8$ channel uses. Thus, we require a constellation of at least $L = 2^{RT} = 256$ signals, each an $M \times T$ matrix, for $M = 1$, 2, and 3.

The following constellations were used in the simulations:

L	δ	K	q	rows of U' (parity)
8	0.000000	1	8	[3 7 6 5 0 4 2]
16	0.306186	1	16	[0 3 14 15 11 10 8]
64	0.353553	3	4	[2 3 3 3 0]
				[2 0 3 1 1]
				[0 3 2 3 3]
133	0.534026	1	133	[48 98 104 72 38 123 4]
256	0.559017	4	4	[1 0 3 1]
				[3 1 1 2]
				[2 0 2 3]
				[1 1 3 2]
529	0.643485	2	23	[14 15 5 5 2 9]
				[11 2 11 4 13 19]
1296	0.695971	4	6	[2 0 1 5]
				[2 5 5 2]
				[2 3 0 3]
				[5 4 2 2]
2209	0.749396	2	47	[20 4 36 43 8 42]
				[44 8 34 6 12 1]

Table 1: Table of best found $M = 1$ antenna constellations of length $T = 8$ based on linear block codes over R_q. The number of signals in the constellation is L, the maximum correlation is δ, the dimension of the block code is K, the arithmetic base is q, and the rows of the parity matrix U' are given last. Note that $L = q^K$.

L	δ	K	q	rows of U' (parity)
4	0.000000	2	2	[0 1 1 0 0 1]
				[0 1 0 1 0 1]
8	0.383533	1	8	[3 0 7 2 5 6 7]
17	0.475099	1	17	[12 11 9 14 6 10 0]
32	0.531944	1	32	[18 11 2 22 8 0 5]
67	0.588905	1	67	[7 31 15 3 29 20 0]
130	0.636015	1	130	[30 71 39 15 4 41 124]
257	0.669317	1	257	[7 60 79 187 125 198 154]
529	0.733934	2	23	[15 3 10 9 15 17]
				[22 16 14 4 21 21]
1024	0.76227	2	32	[26 22 1 3 7 26]
				[18 28 22 8 24 1]
2304	0.803542	2	48	[15 22 27 34 24 41]
				[18 1 38 29 33 25]

Table 2: Table of best found $M = 2$ antenna constellations of length $T = 8$ based on linear block codes over R_q. The number of signals in the constellation is L, the maximum correlation norm is δ, the dimension of the block code is K, the arithmetic base is q, and the rows of the parity matrix U' are given last. Note that $L = q^K$.

- $M = 1$: The $L = 256$ constellation in Table 1.
- $M = 2$: The first 256 signals from the $L = 257$ constellation in Table 2.
- $M = 3$: The first 256 signals from an $L = 257$ constellation where $u = [220\ 191\ 6\ 87\ 219\ 236\ 173\ 170]$ and Φ_1 comprises the first, sixth, and seventh columns of an 8×8 DFT matrix:

$$\Phi_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{8}5} & e^{i\frac{2\pi}{8}6} \\ 1 & e^{i\frac{2\pi}{8}2} & e^{i\frac{2\pi}{8}4} \\ 1 & e^{i\frac{2\pi}{8}7} & e^{i\frac{2\pi}{8}2} \\ 1 & e^{i\frac{2\pi}{8}4} & 1 \\ 1 & e^{i\frac{2\pi}{8}1} & e^{i\frac{2\pi}{8}6} \\ 1 & e^{i\frac{2\pi}{8}6} & e^{i\frac{2\pi}{8}4} \\ 1 & e^{i\frac{2\pi}{8}3} & e^{i\frac{2\pi}{8}2} \end{bmatrix}.$$

  Here $\delta = 0.74355150$.

Figures 4 and 5 show the block and bit error rate for the signal constellations designed for $M = 1$, 2, and 3 transmitter antennas. We see that the bit error rate for larger M drops dramatically as the SNR ρ increases. This is consistent with conclusions of [6] that unitary space-time signals are asymptotically optimal as $\rho \to \infty$.

6 Conclusions

Unitary space-time modulation requires the design of relatively large constellations of matrix valued signals according to a criterion that differs markedly from the traditional maximum Euclidean distance criterion. We have introduced new design algorithms that easily produce large constellations of these signals in a systematic manner. Although not absolutely optimal—and we believe that absolute optimality is virtually impossible to attain for even moderately large constellations—this systematic construction generates unitary space-time signals that exploit transmit diversity in Rayleigh flat fading environments, without either the receiver or the transmitter knowing the fading coefficients.

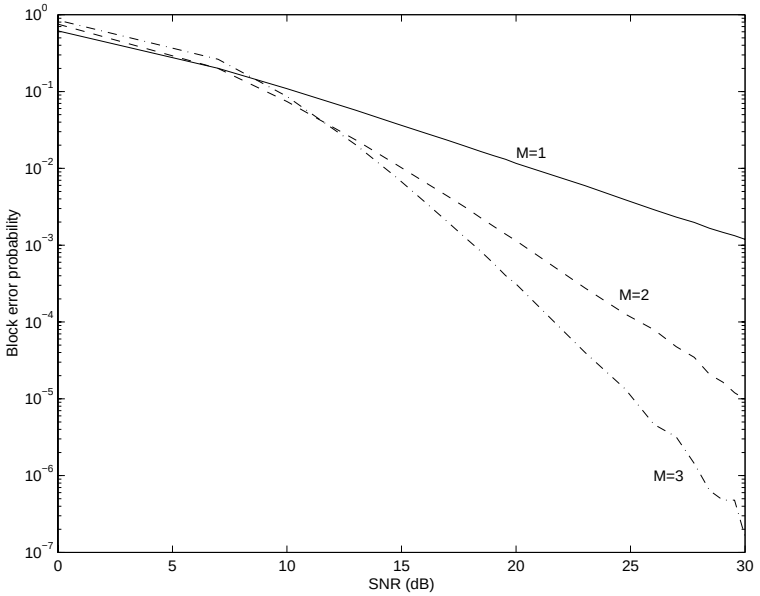


Figure 4: Block error rate for $M = 1$, 2, and 3 transmitter antennas versus SNR with $N = 1$ receiver antenna, $T = 8$, and $R = 1$ bit/channel use.

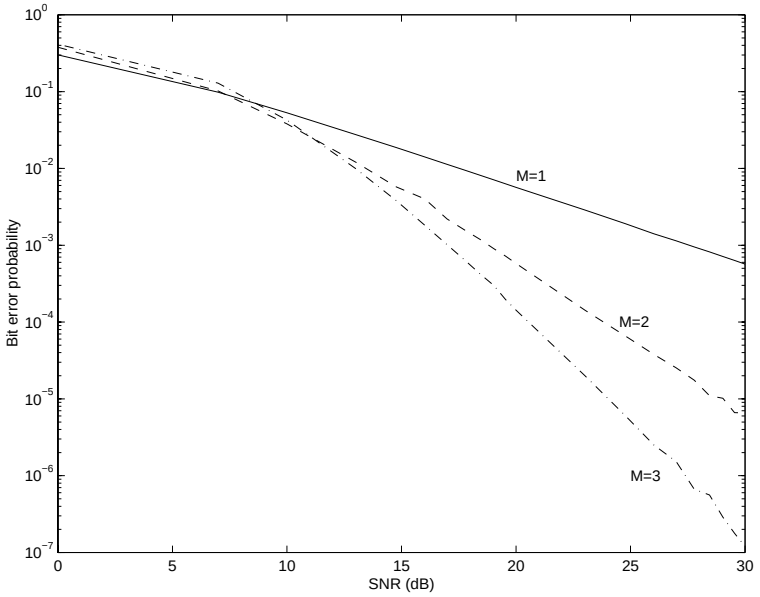


Figure 5: Bit error rate for $M = 1$, 2, and 3 transmitter antennas versus SNR with $N = 1$ receiver antenna, $T = 8$, and $R = 1$ bit/channel use.

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Appendix: Alternate multiple antenna construction

In this appendix we present another method for designing multiple antenna constellations. The basic idea is to rearrange the vectors of a single antenna constellation into a multiple antenna one. As in the beginning of Section 3.1, we take $u_t = t - 1$ for $M = 1$ transmitter antenna. To avoid later confusion, we denote the one antenna signals here with φ_p for $1 \le p \le P$:

$$\varphi_p = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{P}(p-1)} \\ e^{i\frac{2\pi}{P}2(p-1)} \\ \vdots \\ e^{i\frac{2\pi}{P}(T-1)(p-1)} \end{bmatrix}, \qquad p = 1, \ldots, P. \tag{24}$$

As we already know, this results in a sinc-like correlation structure:

$$\left|\varphi_1^\dagger \varphi_p\right| = \begin{cases} 1 & (p = 1) \\ \frac{1}{T}\left|\sum_{t=1}^{T} e^{i\frac{2\pi}{P}(t-1)(p-1)}\right| = \left|\frac{\sin\left(\pi(p-1)T/P\right)}{T\sin\left(\pi(p-1)/P\right)}\right| & (p \neq 1). \end{cases}$$

The vectors φ_p are obtained by projecting down the columns of a $P \times P$ unitary DFT matrix into T dimensions by simply retaining the first T components. This projection clearly ruins the orthogonality of the original DFT columns. However, it is possible to use projections which preserve some orthogonality. Then the columns which remain orthogonal after projection can be used for multiple antennas by making them the columns of the $T \times M$ signal matrices.

If T divides P, some orthogonality is preserved after projection because the sinc-like correlation structure has $T - 1$ zeros. More precisely, for any p the T vectors

$$\left\{\varphi_p,\, \varphi_{(p+P/T) \bmod L} \cdots, \varphi_{(p+(T-1)P/T) \bmod P}\right\},$$

are orthogonal to one another.

For example, with $M = 2$ transmitter antennas and even T, one can construct $L = P/2$ signals, each a $T \times 2$ matrix, by combining the orthogonal vectors φ_p and $\varphi_{p+P/2}$. In general, when M

divides T (and thus P), $L = P/M$ signals can be built as

$$\Phi_l = \left[\varphi_l \;\; \varphi_{l+L} \;\; \cdots \;\; \varphi_{l+(M-1)L}\right], \quad l = 1, \cdots, L,$$

where φ_p is given by (24). One can now show that the singular values of the correlation product $\Phi_\ell^\dagger \Phi_{\ell'}$ are equal and given by

$$d_1 = d_2 = \cdots = d_M = \left\| \Phi_\ell^\dagger \Phi_{\ell'} \right\| = \frac{M}{T} \left| \frac{\sin\left(\pi(l'-l)T/P\right)}{\sin\left(\pi(l'-l)M/P\right)} \right|.$$

Hence, remarkably, we find the same sinc-like behavior as in the one antenna case. For example if $P = 128$, $T = 12$, and $M = 2$, we can construct $L = P/M = 64$ signals, each a 12×2 matrix, whose correlation structure is *exactly* given by Figure 2.

However, as we note in the single-antenna case, this sinc-like correlation is very high when $\ell' = \ell + 1$. Unlike the single-antenna case, the maximum correlation cannot easily be reduced by choosing arbitrary $u_1, \ldots, u_T$, as the needed zeros in the correlation that give us the orthogonal columns are ruined. Thus, more research is needed to develop multiple-antenna signals with low probability of error using this design method.